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RELATIVISTIC UNIFORM CONTINUA

BY

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled RELATIVISTIC UNIFORM CONTINUA submitted by PAUL A. CARLSON in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematical Physics.

ABSTRACT

A new approach is given by the author in this presentation to solve the relativistic problem of continuous media including thermodynamics and heat flow effects. The kinematics of a materially uniform simple body, a concept developed and explained by W. Noll and C.C. Wang is extended to include Lorentz and Galilei space-times. A mathematical formalism for group structures on a manifold is developed, which lends itself well to the description of vector bundles and fibre bundles both for the natural group-reduced tangent bundles for space-time and the corresponding principal bundles. It also provides the basic setting for the framework of material uniformity, a concept extended here to general relativity with the use of these bundles and the material connection.

Using the notion of frame components which arises naturally in the discussion of group structures, a powerful method, similar to the Newman-Penrose formalism, is developed for solving the Einstein field equations in General Relativity. By imposing the Jacobi identity and integrability conditions on the Ricci rotation coefficients it is possible to find solutions that can be interpreted physically in terms of acceleration, expansion rate, rotation etc. Complete formulas are given in adapted frame components for the omnidirectional and unidirectional space-times. A few more general solutions are compared with existing exact ones.

A general theory is then given for a thermodynamic material element, similar in style and theory to that of Noll[1] but extended to include

[1] Noll, W., Arch. Rat. Mech. Anal. 48, 1-50(1972).

thermal effects and done in the formalism of differential geometry for differential type materials. This includes memory effects, and the construction of a materially uniform body as a manifold with a group structure with the symmetry group of the material element. Electromagnetic effects are also examined. The relations to earlier results including the motion in a space-time and the laws of thermodynamics are discussed.

Finally some examples are covered, including the ideal gas and degenerate Fermi gas which show how under particular assumptions on the constitutive equations, classified according to the general theory above, we can see how the Einstein equations, solved in frame components, relate to the material and thermodynamic properties. The possibility of solutions for particular motion types such as constant acceleration, viscometric flow, irrotational, geodesic and isochoric are examined.

TABLE OF CONTENTS

CHAPTER Page

INTRODUCTION AND PRELIMINARIES

(P.1) Summary of Sections.

I have endeavoured in this research to build upon and help establish the formalism for relativistic continuum physics. As a result the work contains many formal definitions and a notation system which, through practice, can be manipulated easily to obtain quick results. A few preliminary remarks on some of the objectives, notation, and background for the chapters are certainly in order here.

In the opening chapter, the basic theory of Classical continuum mechanics [105, 106, 78,79] is extended to the notation and formalism of general relativity. (A few brief introductory remarks on this theory, along the lines of Truesdell is given later in this section.) This includes deformation gradient, mass measure, placement, motion, history, process and deformation. The classical stress tensor is used to construct a stress tensor on space-time, and the existence of this is shown to be equivalent to the classical principle of material frame indifference. The rotation of a material medium, described by a space-time line bundle over the body manifold, is discussed using Fermi-Walker transport by an arbitrary metric connection. Frames along a world line which are spatially and materially nonrotating are discussed, and a number of different connections and corresponding contorsions are used to describe the expansion and rotation properties along lines that are not necessarily world lines. A kinematical description of properties such as vorticity, rotation, deformation rate, expansion rate, density, mass, volume and acceleration is included, with direct comparisons to

classical continuum physics in the original notation used by the authors.

A simple description is given of Lie transport, and the convective derivative [13] is detailed along with the volume elements on the body B and space time M with their convective derivatives. The deformation rate $\theta_{ab} = u_{(a:b)} = u_{(a;b)} + \dot{u}_{(a}u_{b)}$ is directly shown to equal the lift to space-time of the classical time deformation rate D of Truesdell [105]. The referenced and relative left and right Cauchy-Green tensors are completely explained, including transformation properties in their relativistic counterparts. The notion of vector transport using a connection is described and applied to explain kinematic and dynamic properties of space time.

The most important new development in Chapter I is the extension of material uniformity for a simple body of Wang [108] to general relativity. The material connection on the body B which may not be unique but always exists locally, at least, (as Wang has proved) is lifted through the projection to a non-metric material connection on M. The tensor difference between this connection and the Levi-Civita connection on M (referred to as the Christoffel symbols) gives us a contorsion tensor called the material contorsion. By taking symmetric and antisymmetric parts we can describe the expansion and rotation of the material medium in general along any curve in space-time. Associated tensors like the orthogonal delta tensor and generalized rotation are defined and related to the contorsion. These turn out to be very useful in later calculations, particularly in Chapter IV for extending Noll's New Theory of Simple Materials to include electromagnetic effects. The study of the material derivative includes evaluation of the material derivatives of the volume elements, the metric, and the mixed projection tensor.

In Chapter II we bring in the formal mathematics to discuss group structures on a manifold and connections on a group structure. Here we see the formal mathematical concept of a connection discussed by Wang [108] and Kobayashi and Nomizu [44] is brought into direct agreement with the coordinate formulas used in Chapter I and as an added benefit we obtain the results for tensors and connections in a frame component system, in particular, and of greatest interest, in those frame components consistent with the group structure. A great deal of detail is included and this for two basic reasons. Firstly, the abstract presentations in the existing literature are not well suited to relativistic continuum theory, or explained in such a way as to be easily understood by most physicists. Secondly, the theory of group structures on a manifold is the very heart and core of the basics of bundle theory (reduced tangent and principal bundles) and can be used to describe both the Galilei or Lorentz structure on space-time as well as the material uniformity on the body manifold using the symmetry group. The entire structure on the manifold is reduced, through structure preserving isomorphisms to a group structure on a single vector space V, which is a Lie subgroup of the general linear group $GL(V)$. The Lorentz and Galilei structures on V are discussed in some detail including many of their properties, with a number of propositions, remarks and proofs. Later in the chapter, the invariant tensors on the manifold and types of structure preserving connections are covered.

In Chapter III we develop the concepts of frame components associated with a fixed basis of V through structure preserving isomorphisms for a Lorentz group structure. We refer to the usual manifold coordinates as homogeneous coordinates, and to the structure induced frame components as coordinates with torsion. The reason for this naming was made clear

in Chapter II when we derived the formula for the torsion of a connection in frame components. The "coordinate torsion" turns out to be the Ricci coefficients for the frame component system. They satisfy the Jacobi identity $T_{[j\ k,\ell]}^{\ i} + T_{a\ [\ell}^{\ i}T_{j\ k]}^{\ a} = 0$ and the integrability condition $T_{j\ k,\ell m}^{\ i} - T_{j\ k,m\ell}^{\ i} = T_{\ell\ m}^{\ a}T_{j\ k,a}^{\ i}$. The Christoffel symbols in metric frame components are given by $\left\{ {i \atop j\ k} \right\} = \frac{1}{2}(T_{j\ k}^{\ i} - T^{i}_{\ jk} - T^{i}_{\ kj})$ where $(\eta_{jk}) =$ diag(1,1,1,-1) is used to raise and lower indices. This technique, which has similarities to the Newman-Penrose formalism, is useful in relativistic continuum theory because all dynamical quantities and kinematic rate quantities can be expressed in terms of the Ricci coefficients, without reference to the local orthonormal tetrad field defining them. Symmetry conditions that reduce the number of solutions to the point of making the equations of General Relativity explicitly soluble can be imposed. The ones considered of greatest interest, namely omni- and unidirectionality have complete formulas given. The local orthonormal frame used is said to be adapted, in that the time like unit vector field is the matter flow, and the three orthonormal space like vector fields are principal stress vectors. Under these circumstances we have the deformation rate $\theta_{AB} = -\frac{1}{2}(T^{4}_{\ BA} + T^{4}_{\ AB})$, rotation rate $\omega_{AB} = \frac{1}{2}T_{A\ B}^{\ 4}$, acceleration $\dot{u}_a = T_{a\ 4}^{\ 4}$ and expansion rate $\theta = T_4 = T_{4\ a}^{\ a}$, where $a = 1,2,3,4$, $A,B = 1,2,3$ and the components are given in the adapted frame. By the Einstein equations, the orthogonal part of the Ricci tensor is diagonal in this frame, and it is given by $R_{jk} = T_{(j,k)} + T^{i}_{\ (jk),i} + \frac{1}{2}B_{jk} - \frac{1}{4}E_{jk} + A_{(jk)} + \frac{1}{2}G_{jk}$ where $B_{jk} = T^{i}_{\ aj}T_{i\ k}^{\ a}$, $E_{jk} = T_{aj}^{\ \ i}T^{a}_{\ ki}$, $A_{jk} = T_aT^{a}_{\ jk}$ and $G_{jk} = T_{a\ k}^{\ i}T_{i\ j}^{\ a}$ where $T_a = T_{a\ i}^{\ i}$. Exact solutions to the field equations are obtained in simple cases, which are of interest in cosmology, and more general cases, which are examined in the fifth and final chapter on special

solutions in the light of the formalism developed here.

In the fourth chapter, the New Theory of Simple Materials of Noll [77] is extended to relativity and thermodynamics with some modifications. The state space Σ of Noll is a differential manifold here, and as a consequence does not have the same topology as he introduces. The evolution map $\hat{\rho}$ is different, adapted more to the formalism of differential geometry in Dieudonné [22]. Thus the properties of the response functional in continuum mechanics are incorporated in the smooth maps $\hat{\rho}: E\Sigma \to T\Sigma$ and $\hat{S} : \Sigma \to \mathcal{S}$, where $E\Sigma$ is the evolution bundle and $\hat{S}$ is the stress map, $\mathcal{S}$ the stress space. The structure of a material element on an n-dimensional vector space is completely given along with all the axioms, and we define what it means for two material elements to be materially isomorphic. The symmetry group of all material isomorphisms of a material element with itself is introduced, and in the formalism of group structures on a manifold, the body B is given the symmetry group structure. The coordinate notation of space time is related to the body element formulation. Thermodynamics is included, and the first and second laws are studied in detail. The Maxwell equations of electromagnetic theory are reduced from space-time to the body, and examined in the light of this abstract continuum theory with memory, although a full solution to this problem seems distant.

In Chapter V we take the special solutions of Chapter III and the material classifications using the formalism of Chapter IV and the special motion types of Chapter I and put them together. Special solutions are found for material media in general relativity. Some of the constitutive equations of kinetic gas theory and Boltzmann temperature determined energy distributions are considered.

(P.2) Symbols and Notation.

Some of the notation and terminology used in this work should be considered now. There are a few minor changes from one chapter to another but I have attempted to keep these to a minimum. The symbol M is used to denote space-time, B the body manifold, $P : M \to B$ the projection. The metric in coordinates is g_{ab}, the flow vector is u^a with $u^a u_a = -1$, and $\gamma_{ab} = g_{ab} + u_a u_b$ is the orthogonal metric. The Levi-Civita connection (or Christoffel symbols) is denoted by $\left\{{a \atop b\ c}\right\} = \frac{1}{2} g^{ad} \{g_{db,c} + g_{dc,b} - g_{bc,d}\}$. A semicolon is used to denote the covariant differentiation with respect to this connection, so, for instance $u^a{}_{;b} = u^a{}_{,b} + u^c\left\{{a \atop c\ b}\right\}$ and $u_{a;b} = u_{a,b} - u_c\left\{{c \atop a\ b}\right\}$. The comma is used to indicate partial differentiations in the usual homogeneous coordinates, and canonical parametric derivative along the indicated integral curve of the specified vector field in frame components, which simply extends its definition naturally. If another connection is given, say $\Gamma_b{}^a{}_c$ in components, we write $\Gamma_b{}^a{}_c = \left\{{a \atop b\ c}\right\} - K_b{}^a{}_c$ and call $K_b{}^a{}_c$ the *contorsion tensor* for $\Gamma_b{}^a{}_c$. We associate the contorsion with the connection and the corresponding covariant differentiation, some examples of which are shown below. $(\dot{u}_a = u_{a;b}u^b)$

Connection		Covariant Differentiation	Contorsion
Christoffel Symbol	$\left\{{a \atop b\ c}\right\}$	$;$	$\overset{;}{K}_b{}^a{}_c = 0$
Colon	$\Gamma_b{}^a{}_c = \overset{:}{\Gamma}_b{}^a{}_c$	$:$	$\overset{:}{K}_b{}^a{}_c = (u_b \dot{u}^a - u^a \dot{u}_b)u_c$
	$\overset{!}{\Gamma}_b{}^a{}_c$	$!$	$\overset{!}{K}_{bca} = u_{[b,c]}u_a + u_{[a,c]}u_b + u_{[b,a]}u_c$
	$\overset{\dot{\vert}}{\Gamma}_b{}^a{}_c$	$\dot{\vert}$	$\overset{\dot{\vert}}{K}_{abc} = (u_{[a;b]} + u_{[a}\dot{u}_{b]})u_c$

	Connection	Covariant Differentiation	Contorsion
Dot	$\overset{\bullet}{\Gamma}{}^{a}_{b\ c}$	$\cdot$	$\overset{\bullet}{K}_{acb} = u_c u_{a;b} - u_a u_{c;b}$
Star	$\overset{*}{\Gamma}{}^{a}_{b\ c}$	$*$	$\overset{*}{K}_{acb} = 2u_{[c}u_{a];b} + \omega_{ac}u_b$
Material	$\hat{\Gamma}{}^{a}_{b\ c}$	$\hat{}$	$\hat{K}_{abc} = \hat{\Delta}_{abc} + u_{a;c}u_b - u_{b;c}u_a - u_{b;a}u_c - \dot{u}_b u_a u_c$
Funda-mental	$\Gamma^{a}_{b\ c}$	$\mid$	$K_{abc} = \Delta_{abc} + u_{a;c}u_b - u_{b;c}u_a - u_c\omega_{ba}$

Of course $u_{a:b} = u_{a;b} + \dot{u}_a u_b$ and we put $\theta_{ab} = u_{(a:b)}$ and $\omega_{ab} = u_{[a:b]}$. We use $\mathcal{D}$ for the convective derivative, and P^{α}_{a} for the components of the mixed projection tensor. For indices, the Latin a, b, i, j etc. run from 1 to 4 referring to space time and the Greek α, β run from 1 to 3 and refer to the body.

Some of the notation of Noll [77] which is used to quite an extent in Chapter IV is introduced earlier, especially in Chapter I. We briefly review some of it at this time. If $\mathcal{T}$, $\mathcal{T}_1$ and $\mathcal{T}_2$ are n-dimensional real vector spaces, then $\mathrm{Lin}(\mathcal{T}_1,\mathcal{T}_2)$ is the set of all linear maps from $\mathcal{T}_1$ to $\mathcal{T}_2$, and $\mathrm{Invlin}(\mathcal{T}_1,\mathcal{T}_2)$ is the set of all linear isomorphisms (if $\dim \mathcal{T}_1 = \dim \mathcal{T}_2$). We let $\mathrm{Lin}(\mathcal{T}) = \mathrm{Lin}(\mathcal{T},\mathcal{T})$ and $GL(\mathcal{T}) = \mathrm{Invlin}(\mathcal{T},\mathcal{T})$, the latter being the linear automorphisms. If $\mathcal{T}^*$ is the dual of $\mathcal{T}$ then $\mathrm{Sym}(\mathcal{T},\mathcal{T}^*)$ consists of those linear maps from $\mathcal{T}$ to $\mathcal{T}^*$ which canonically correspond to a symmetric bilinear form on $\mathcal{T}$. Likewise $\mathrm{Sym}^+(\mathcal{T},\mathcal{T}^*)$ contains maps to be identified with positive definite symmetric bilinear forms on $\mathcal{T}$. If $I \in \mathrm{Sym}^+(\mathcal{T},\mathcal{T}^*)$ then $\mathcal{O}(I)$, the orthogonal group of I is the set of all elements g in $GL(\mathcal{T})$ with $g^* \circ I \circ g = I$.

(P.3) Continuum Mechanics - Classical Kinematics.

Finally, let us examine some of the classical continuum theory of Truesdell, Noll and Wang. A body (material medium or continuum) is a 3-dimensional connected differential manifold B that can be covered by a single coordinate chart, i.e. is diffeomorphic to an open subset of $\mathbb{R}^3$. The coordinate charts $\{(U_\alpha,\phi_\alpha), \alpha \in A\}$ with $U_\alpha = B$ are of greatest interest and we call each such diffeomorphism $\phi_\alpha : B \to \phi_\alpha(B) \subset \mathbb{R}^3$ a *global placement* of B in $\mathbb{R}^3$, where $\phi_\alpha(B)$ is an open connected subset of $\mathbb{R}^3$. Any other such $\phi_\alpha : U_\alpha \to \mathbb{R}^3$ is a *local placement.* The derivative map $\phi_{\alpha*X}: B_X \to \mathbb{R}^3$ defines an *(infinitesimal) placement of* X which is simply a linear isomorphism that preserves the standard orientation, where B_X is the tangent space to B at $X \in B$. We have $\phi^*_{\alpha X}: \mathbb{R}^3 \to B^*_X$ which is the dual isomorphism of the one above, and the natural inner product on $\mathbb{R}^3$ is used to equate $\mathbb{R}^3$ with $\mathbb{R}^{3*}$. The map $\phi^*_{\alpha X} \circ \phi_{\alpha*X} \in \mathrm{Sym}^+(B_X,B^*_X)$ can be viewed as a positive definite symmetric bilinear form on B_X called the *(infinitesimal) configuration* of X in the placement determined by ϕ_α, where $X \in U_\alpha$. If $\phi_\alpha: U_\alpha \to \mathbb{R}^3$ is a local placement, there is a unique structure (U_α,d_α) of a metric space on U_α which makes ϕ_α an isometry. We call d_α the *local configuration* of U_α under the placement ϕ_α, and it measures faithfully the stretching of that portion of the material medium or body described by the open set U_α. Similarly we can define *global configuration.* The placement determines the configuration (or deformation) of the material completely, whether local, global or infinitesimal, but the converse is not true, if we consider rigid rotations or translations for instance.

The body manifold B is assumed to be equipped with a σ-finite mass measure m on Borel sets of B which is absolutely continuous

in each local configuration with respect to the volume Lebesgue measure dV in $\mathbb{R}^3$ with Radon-Nikodym derivative ρ called the *mass density*, i.e. for $A \subset U_\alpha$, $m(A) = \int_{\phi_\alpha(A)} \rho_\alpha dV$, and ρ_α is a real valued function on $\phi_\alpha(A) \subset \mathbb{R}^3$. Its transformation properties under a change of placement will be given in (I.3).

It is worth noting at this time we are using the more modern term "placement" [77, p. 8] in place of the original term "configuration" [105, p. 5]. A finer distinction will be made in Chapter IV between configuration and deformation which are equivalent in this chapter.

A *(local) reference placement* $r: U \to \mathbb{R}^3$ is a distinguished local placement. Similarly we have global and infinitesimal reference placements. We call r_{*X} the *infinitesimal reference placement at* X *determined by* r for $X \in U$. If $\phi: U \to \mathbb{R}^3$ is a placement, we call $\phi \circ r^{-1}: r(U) \subset \mathbb{R}^3 \to \phi(U) \subset \mathbb{R}^3$ the *related local placement*. Similar definitions exist for the related global and infinitesimal placements. The classical "deformation gradient" [105, p. 11] is the linear automorphism $\text{Grad}\, \phi \circ r^{-1}|_{r(X)}$ of $\mathbb{R}^3$ where $X \in B$, and we write $r(X) = (X^1, X^2, X^3)$ as the "components" of X.

A *local motion* is a smooth time parametrized family of local placements, i.e. $\phi_t: U \to \mathbb{R}^3$ is a diffeomorphism onto an open subset of $\mathbb{R}^3$ for each t, and $t \in I \subset \mathbb{R}$ where I is an interval (open closed, bounded or unbounded). If $I = [0, d_p]$ where $d_p > 0$, and $\phi_t = P(t): U \to \mathbb{R}^3$ is a local motion we call P a *local process* and d_p its *duration*. If $I = (-\infty, t_0]$ we call $\phi_t = H(t): U \to \mathbb{R}^3$ a *local history* with *limit* t_0. Similar definitions hold for global and infinitesimal motions, processes and histories.

A *local deformation motion* corresponding to the local motion (or

local placement motion) $\phi_t\colon U \to \mathbb{R}^3$, $t \in I$ is the map $t \longrightarrow d_t$ where d_t is the metric on U making ϕ_t an isometry. An *infinitesimal deformation motion* corresponding to the infinitesimal motion $t \longrightarrow \kappa_t\colon B_X \to \mathbb{R}^3$ is the map $t \longrightarrow \kappa_t^* \circ \kappa_t \in \mathrm{Sym}(B_X, B_X^*)$ for $t \in I$. By appropriate restrictions of I as above we also have the local deformation process, local deformation history, infinitesimal deformation process and infinitesimal deformation history. By putting $U = B$ above we have a global deformation motion, process or history.

If ϕ_t, $t \in I$ is a local motion and $t_0 \in I$ is selected, we may put $r\colon U \to \mathbb{R}^3$ equal to ϕ_{t_0}. Then $\phi_t \circ r^{-1}\colon r(U) \subset \mathbb{R}^3 \to \phi_t(U) \subset \mathbb{R}^3$ is called the *relative local motion* [105, p. 15, 16]. Relative global and infinitesimal motions (for placements) can also be defined, but deformations are not relative in this sense. The "relative deformation gradient" [105, p. 16] is the gradient of the relative local motion, an automorphism of $\mathbb{R}^3$ at each point $(X^1, X^2, X^3) \in r^{-1}(U) \subset \mathbb{R}^3$, and a function of $t \in I \subset \mathbb{R}$. We use Truesdell's term *deformation gradient* to refer to the gradient of either a fixed related local placement, or a *related local motion* $\phi_t \circ r^{-1}$ where $r\colon U \to \mathbb{R}^3$ is a fixed local reference placement, not necessarily equal to any particular ϕ_{t_0}. In this latter case it is a function of t, while in the former it is not.

If $F\colon \mathbb{R}^3 \to \mathbb{R}^3$ is a linear automorphism which is a deformation gradient we write $F = RU = VR$ where R is orthogonal and U and V are positive definite symmetric. This is the unique Cauchy decomposition [105, p. 17]. Because of our orientation requirements, $\det F > 0$ so $\det R = +1$. We call R the *rotation*, U and V the *right and left stretches*, and $C = U^2 = F^T F$, and $B = V^2 = FF^T$ the[1] *right and left*

---

[1] The map $F^T\colon \mathbb{R}^3 \to \mathbb{R}^3$ is the dual isomorphism $F^*\colon \mathbb{R}^{3*} \to \mathbb{R}^{3*}$ with the canonical inner product on $\mathbb{R}^3$ used to identify $\mathbb{R}^3$ with $\mathbb{R}^{3*}$.

Cauchy-Green tensors respectively. For the corresponding related tensors on $\mathbb{R}^3$, we leave them unsubscripted, but for a relative deformation gradient F_{t_0} or $F_{t_0}(t)$ we subscript the reference time. Hence we obtain $R_{t_0}(t)$, $U_{t_0}(t)$, V_{t_0}, C_{t_0}, B_{t_0} etc. This notation is used in (I.12).

Following Truesdell [105, p. 19] (see also (I.12)) we define the instantaneous time derivative of the relative deformation gradient by $\dot{F}_t(t) \equiv \frac{d}{d\tau} F_t(\tau)\big|_{\tau=t}$. We put $G = \dot{F}_t(t)$, $D = \dot{U}_t(t) = \dot{V}_t(t)$, $W = \dot{R}_t(t)$. Then [105, p. 19,20], $D^T = D$, $W^T + W = 0$, $G = \dot{F}(t)F(t)^{-1} = D + W$. Here $F(t)$ is the deformation gradient at time t with corresponding $U(t) = U$, V, R etc.. $\dot{F}(t)$ is simply the time derivative. It is not hard to show that

$$
\begin{aligned}
W &= \dot{R}R^T + \frac{1}{2} R(\dot{U}U^{-1} - U^{-1}\dot{U})R^T \quad \text{and} \\
D &= \frac{1}{2} R(\dot{U}U^{-1} + U^{-1}\dot{U})R^T \quad [105, \text{ p. } 21].
\end{aligned}
\tag{P.3.1}
$$

It is customary to use (X^1,X^2,X^3,t) as the space-time coordinates in classical theory with the usual time t and components X^1, X^2, X^3 with respect to the canonical basis of $\mathbb{R}^3$. It is possible to consider a change of frame [105, p. 22] that preserves distance, time intervals and time sense, which will be of the form

$$
X^{*i} = C^i(t) + Q^i{}_j(t)(X^j - X_0^j), \quad t^* = t - a,
$$

where $C^i(t)$ are the components in $\mathbb{R}^3$ of a time dependent point, $Q^i{}_j$ is an orthogonal matrix function of t, $\vec{X}_0$ a fixed vector (or point) in $\mathbb{R}^3$ and $a \in \mathbb{R}$ is a constant. We write $(X^{*1},X^{*2},X^{*3},t^*)$ as the coordinates of the same point in the new frame. A *frame indifferent* quantity is invariant under all changes of frame. For instance if A is an indifferent scalar, $A^* = A$, if $\underset{\sim}{v}$ is an indifferent vector,

$\underset{\sim}{v} \in \mathbb{R}^3$, then $\underset{\sim}{v}^* = Q\underset{\sim}{v}$, and if S is an indifferent tensor of second order then $S^* = QSQ^T$. Indifference means the tensor is intrinsic to the body or material medium. One can see easily that for the related deformation gradient and associated quantities, $F^* = QF$, $R^* = QR$, $U^* = U$, $V^* = QVQ^T$, $D^* = QDQ^T$, $W^* = QWQ^T + A$ where $A = \dot{Q}Q^T = -A^T$, (cf. [105, p. 24] also (I.12)). We call A the *angular velocity* of the starred frame with respect to the unstarred one. Thus V and D are frame indifferent, while the others are not.

(P.4) The Stress Tensor.

If $A \subset U \subset B$ is enclosed by its boundary ∂A the forces on A are classified in classical theory as body force $\underset{\sim}{f}_b(A)$ and contact force $\underset{\sim}{f}_c(A)$, the former being an absolutely continuous function of the volume of $\phi_t(A)$, and the latter an absolutely continuous function of the surface area of $\partial\phi_t(A)$, its boundary. Of course $\phi_t\colon U \to \mathbb{R}^3$ is the local motion under consideration, (cf. [105, p. 26] also (I.4)). The resultant force $\underset{\sim}{f}(A)$ acting on A can be written as $\underset{\sim}{f}(A) = \underset{\sim}{f}_b(A) + \underset{\sim}{f}_c(A)$ where $\underset{\sim}{f}_b(A) = \int_{\phi_t(A)} \underset{\sim}{b}\rho_t\, dV = \int_A \underset{\sim}{b}\, dm$ and $\underset{\sim}{f}_c(A) = \int_{\partial\phi_\alpha(A)} \underset{\sim}{t}\, dS$. We say that $\underset{\sim}{b}$ is the *specific body force* and $\underset{\sim}{t}$ is the *traction,* these being indifferent vectors, i.e. $\underset{\sim}{b}^* = Q\underset{\sim}{b}$ and $\underset{\sim}{t}^* = Q\underset{\sim}{t}$ under a change of frame. Translating the mass measure over to $\phi_t(U)$ using the diffeomorphism ϕ_t we may write $\underset{\sim}{f}_b(A) = \int_{\phi_t(A)} \underset{\sim}{b}\, dm$.

Using this notation we define the *resultant moment of force*

$$\underset{\sim}{L}(A,\underset{\sim}{x}_0) = \int_{\phi_t(A)} (\underset{\sim}{x} - \underset{\sim}{x}_0) \times \underset{\sim}{b}\ dm + \int_{\partial\phi_t(A)} (\underset{\sim}{x} - \underset{\sim}{x}_0) \times \underset{\sim}{t}\ dS, \tag{P.4.1}$$

where dS is the surface area element on $\partial\phi_t(A)$ as before, $x_0 \in \mathbb{R}^3$ is a fixed point, and $x \in \mathbb{R}^3$ is the evaluation point for the integra-

tion done at a fixed time t (classically), and "$\times$" is the three dimensional vector cross product, (cf. [105, p. 27]). The moments of force about different points are related by $\underset{\sim}{L}(A,\underset{\sim}{x}_0') = \underset{\sim}{L}(A,\underset{\sim}{x}_0) + (\underset{\sim}{x}_0 - \underset{\sim}{x}_0') \times \underset{\sim}{f}(A)$. If $X \in U \subset B$ is fixed, we define $\dot{\underset{\sim}{x}} = \left(\frac{dX^1}{dt}, \frac{dX^2}{dt}, \frac{dX^3}{dt}\right)$ where $(X^1,X^2,X^3) = \phi_t(X)$. We then have the *momentum* $\underset{\sim}{m}(A)$ and *angular momentum* $\underset{\sim}{M}(A,\underset{\sim}{x}_0)$ of A in the placement ϕ_t given by

$$\underset{\sim}{m}(A) = \int_{\phi_t(A)} \dot{\underset{\sim}{x}}\, d\,m, \quad \underset{\sim}{M}(A,\underset{\sim}{x}_0) = \int_{\phi_t(A)} (\underset{\sim}{x} - \underset{\sim}{x}_0) \times \dot{\underset{\sim}{x}}\, d\,m. \qquad \text{(P.4.2)}$$

We then have *Euler's Laws of Mechanics*, given by $\underset{\sim}{f}(A) = \dot{\underset{\sim}{m}}(A)$ and $\underset{\sim}{L}(A,\underset{\sim}{x}_0) = \dot{\underset{\sim}{M}}(A,\underset{\sim}{x}_0)$, where the dot indicates time derivative.

According to Cauchy's Fundamental Lemma [105, p. 29] (a stronger result that follows from the Euler-Cauchy Stress Principle) and the Euler Laws of Mechanics, we may write the traction as $\underset{\sim}{t}(\underset{\sim}{x}, t, \underset{\sim}{n}) = T(\underset{\sim}{x}, t)\underset{\sim}{n}$, where $\underset{\sim}{n}$ is the unit normal in $\mathbb{R}^3$ to $\partial\phi_t(A)$ and T is the *stress tensor*. Using the divergence theorem we can see that Euler's Laws of Mechanics [105, p. 31] can be written in an equivalent form as Cauchy's laws of continuum mechanics, namely $\rho\ddot{\underset{\sim}{x}} = \operatorname{div} \underset{\sim}{T} + \rho\underset{\sim}{b}$ and $T^T = T$. Specific assumptions involved in this equivalence are that all torques are moments of forces, and that the traction is simple, i.e. it satisfies the Euler-Cauchy stress principle $\underset{\sim}{t} = \underset{\sim}{t}(\underset{\sim}{x}, t, \underset{\sim}{n})$. They express locally the balance of linear momentum and moment of momentum (or angular momentum).

(P.5) Constitutive Equations and Simple Materials.

The Constitutive Equation determines the stress tensor as a function of the placement and deformation of the body in past time. Several basic principles which govern the formulation of constitutive equations ought to be stated now (cf. [105, p. 33]).

1. Principle of Determinism: *The stress tensor at* t_0, *(a function defined on* $\phi_{t_0}(B) \subset \mathbb{R}^3$ *for* ϕ_t *a global motion, which is a second order symmetric tensor on* $\mathbb{R}^3$*) is determined by the global history* $H(t) = \phi_t$ *with limit* t_0.

We call the functional of all possible histories with limit t_0 which gives us the stress tensor at t_0 in each case, the *response functional*. The equation which equates the stress tensor to its functional of a history, is called the *constitutive equation*.

2. Principle of Local Action: *The stress tensor at* $\left(\phi_{t_0}(X), t_0\right) = (X_0^1, X_0^2, X_0^3, t_0)$ *is determined by the local history* $H(t) = \phi_t : U \to \mathbb{R}^3$ *for any neighborhood* U *of* $X \in B$, *where* H *has limit* t_0.

3. Principle of Material Frame Indifference: *The stress tensor at* t_0 *on* $\phi_{t_0}(B)$ *for* ϕ_t *a global motion is determined by the global deformation history with limit* t_0.

By combining the principle of local action with material frame indifference we see that the stress tensor at $\left(\phi_{t_0}(X), t_0\right)$ is determined by the local deformation history with limit t_0 of any neighborhood U of X.

We say that B is a *simple body* or is a *simple material* if the stress tensor at $\phi_{t_0}(X) \in \mathbb{R}^3$ at time t_0 is determined by the infinitesimal history of X with limit t_0, for every $X \in B$. Material frame indifference implies we need only know the infinitesimal deformation history of X up to time t_0. Following the notation of Wang [108, p. 39], we write the constitutive equation for the stress as $T\left(\phi_t(X), t\right) = \underset{s=-\infty}{\overset{t}{F}}\left(\phi_{*X}(s), X\right)$. We call F the *response functional* (as above) and

$\phi_{*X}: B_X \to \mathbb{R}^3$ is a linear isomorphism for each X and a function of the time parameter s which is written as a main function variable here rather than a subscript as it was used before. This equation shows the direct dependence of the stress on the infinitesimal history, the response functional also depending on the position X in the body.

Relative to a fixed infinitesimal reference placement $r(X): B_X \to \mathbb{R}^3$ we can write the constitutive equation in the form $T(\phi_t(X),t) = \underset{s=-\infty}{\overset{t}{G}}{}_{r(X)}(F(s),X)$ where G is the *response functional relative to* $r(X)$ [108. p. 39] and $F(s) = \phi_{*X}(s) \circ r(X)^{-1}$ is the deformation gradient history. In other words, if κ is any infinitesimal history of $X \in B$, i.e. $\kappa(s): B_X \to \mathbb{R}^3$, $s \leq t_0$ then

$$\underset{s=-\infty}{\overset{t}{F}}(\kappa(s),X) = \underset{s=-\infty}{\overset{t}{G}}{}_{r(X)}(\kappa(s) \circ r(X)^{-1},X) \quad \text{where}$$

$$\kappa(s) \circ r(X)^{-1}: \mathbb{R}^3 \to \mathbb{R}^3,$$

for $t \leq t_0$.

(P.6) Material Uniformity.

A simple body B consists of simple material particles or points $X \in B$ only. Of course we say that $X \in B$ is a *simple particle* if the stress at $(\phi_{t_0}(X),t_0)$ is determined by the infinitesimal history of X up to t_0. If X and Y are simple particles we say [108, p. 40] they are *materially isomorphic* if

$$\underset{s=-\infty}{\overset{t}{G}}{}_{r(X)}(F(s),X) = \underset{s=-\infty}{\overset{t}{G}}{}_{r(Y)}(F(s),Y) \tag{P.6.1}$$

for some infinitesimal reference placements $r(X): B_X \to \mathbb{R}^3$ and

$r(Y)\colon B_Y \to \mathbb{R}^3$, and for every 3×3 invertible matrix function[2] history F. This is equivalent to the condition

$$\mathop{\mathcal{F}}_{s=-\infty}^{t}\left(\phi_{*Y}(s), Y\right) = \mathop{\mathcal{F}}_{s=-\infty}^{t}\left(\phi_{*Y}(s) \circ r(Y)^{-1} \circ r(X), X\right), \qquad (P.6.2)$$

using the other form of the response functional, for all local histories ϕ_s about Y.

A simple body B is *materially uniform* if for any $X, Y \in B$, X and Y are materially isomorphic. Let B be materially uniform. A *reference chart* for B is a pair (U_α, r_α) where U_α is an open subset of B called a *reference neighborhood* and $r_\alpha\colon X \subset U_\alpha \to r_\alpha(X)$ is a smooth field of infinitesimal reference placements called a *reference map*. We require that for all $X \in U_\alpha$,

$$\mathop{\mathcal{G}}_{s=-\infty}^{t}{}_{r_\alpha(X)}\left(F(s), X\right) = \mathop{\mathcal{G}_\alpha}_{s=-\infty}^{t}\left(F(s)\right),$$

that is the response relative to $r_\alpha(X)$ at X should be independent of $X \in U_\alpha$ but depends on α only. This holds for all histories $F(s)$, $s \leq t$ and defines for us the *response functional* $\mathcal{G}_\alpha$ *of the reference chart* (U_α, r_α).

We say that two reference charts (U_α, r_α) and (U_β, r_β) are *compatible* if the corresponding response functionals $\mathcal{G}_\alpha$ and $\mathcal{G}_\beta$ are identical. This of course is an equivalence relation, independent of the overlap of the charts. A collection $\mathcal{U} = \{(U_\alpha, r_\alpha), \alpha \in \mathcal{B}\}$ of mutually compatible reference charts is a *reference atlas* if it is maximal and if $\{U_\alpha, \alpha \in \mathcal{B}\}$ is an open cover for B. We put $\mathcal{G}_{\mathcal{U}} = \mathcal{G}_\alpha$, $\forall\, \alpha \in \mathcal{B}$ as the *response functional relative to* $\mathcal{U}$.

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[2] We naturally associate a linear map $\mathbb{R}^3 \longrightarrow \mathbb{R}^3$ with its matrix with respect to the canonical basis.

The development of material uniformity for simple bodies is given in detail by Wang [108]. He covers the symmetry groups p. 42 (or isotropy groups in the old terminology) and material connections p. 62, which we will mention briefly here. First of all, we should consider material frame indifference.

(P.7) Material Frame Indifference for a Simple Body.

For a simple material, the condition that the stress depends only on the infinitesimal deformation history tells us that $T(\phi_t(X),t) = \underset{s=-\infty}{\overset{t}{\mathcal{F}}}(\phi_{*X}(s),X)$ can be rewritten as a function of $\phi_X^*(s) \circ \phi_{X*}(s)$. This means that the stress tensor is frame indifferent (P.3) so that the principle of material frame indifference can be expressed (using the change of frame from unstarred to starred) as

$$\underset{s=-\infty}{\overset{t}{\mathcal{F}}}\left(Q(s)\circ\kappa_s,X\right) = Q(t)\left[\underset{s=-\infty}{\overset{t}{\mathcal{F}}}(\kappa_s,X)\right]Q(t)^T, \text{ or}$$
$$\underset{s=-\infty}{\overset{t}{\mathcal{G}_{r(X)}}}\left(Q(s)F(s),X\right) = Q(t)\left[\underset{s=-\infty}{\overset{t}{\mathcal{G}_{r(X)}}}\left(F(s),X\right)\right]Q(t)^T \tag{P.7.1}$$

for all orthogonal tensor histories $Q(t)$, and all deformation gradient histories $F\colon \mathbb{R}^3 \to \mathbb{R}^3$ (linear isomorphism for each t) or equivalently all (infinitesimal) placement histories κ. This form is used in Wang [108, p. 42], Truesdell [105, p. 39] and later is extended to general relativity (I.6).

(P.8) Symmetry Groups and Material Connections.

We let B be a simple body which is materially uniform and smooth. Then any $X,Y \in B$ are materially isomorphic, and we say that $r(Y)^{-1}\circ r(X)\colon B_X \to B_Y$ is a *material isomorphism* if (P.6.1) holds for

all F. The *symmetry group* $G(X)$ of B at X is the group of all material isomorphisms of X with itself. Thus $h \in GL(B_X)$ is an element of $G(X)$ if and only if $\underset{s=-\infty}{\overset{t}{F}}\left(\phi_{*X}(s),X\right) = \underset{s=-\infty}{\overset{t}{F}}\left(\phi_{*X}(s)\circ h,X\right)$ for all local histories ϕ about X, (cf. [108, p. 42]). Following Wang we require that $G(X)$ be a Lie subgroup of $SL(B_X)$ the special linear group of linear automorphisms on B_X, the tangent space to B at $X \in B$. These all have determinant $+1$ and are volume and orientation preserving. Physically this means that the stress changes if the volume occupied per unit mass changes, i.e. we have no volume altering symmetries. Furthermore since our body is materially uniform $G(X)$ and $G(Y)$ are isomorphic Lie Groups for all $X,Y \in B$. Using the structure of our reference atlas, the association $X \longrightarrow G(X)$ becomes a Group Structure on the Differential Manifold B (cf. Chapter II) whose associated group reduced tangent and principal bundles are the material tangent and material principal bundles of Wang [108, pp. 46-62]. This follows from the result due to Noll and Wang that if on $\mathbb{R}^3$ we define a subgroup $G \subset SL(\mathbb{R}^3)$ by $g \in G$ if and only if $\underset{s=-\infty}{\overset{t}{\mathcal{G}_U}}\left(F(s)\right) = \underset{s=-\infty}{\overset{t}{\mathcal{G}_U}}\left(F(s)g\right)$ for all F, then G is a Lie subgroup of $SL(\mathbb{R}^3)$ which is isomorphic to each $G(X) = G_X$. Furthermore if $\mathcal{U} = \{(U_\alpha, r_\alpha),\ \alpha \in \mathcal{B}\}$ is the reference atlas, for $X \in U_\alpha$, $r_{\alpha X}\colon B_X \to \mathbb{R}^3$ induces a Lie group isomorphism $\hat{r}_{\alpha X}\colon G_X \to G$ defined by $\hat{r}_{\alpha X}(g_X) = r_{\alpha X}\circ g_X \circ r_{\alpha X}^{-1}$ for $g_X \in G_X$, $g_X\colon B_X \to B_X$, ([108, p. 43] and (II.1)). This gives us the mathematical formalism needed to describe the notion of a material being uniform in classical continuum theory, with the notion of smoothness handled by the theory of group structures and differential geometry. One of our main objectives will be to extend this to relativity, and look for possible solutions to the field equations there for media which are materially uniform.

In the classical theory presented thus far, Wang [108] has proved quite a number of propositions and theorems relating to particular types of smooth materially uniform media (referred to as "simple bodies" by him). One of the most important concepts he develops is that of the material connection. A material connection is simply a linear connection (Kobayashi and Nomizu [44]) whose parallel transports are material isomorphisms. It is a connection on the group structure (II.6) $\mathcal{G}(B)$ which is the symmetry group structure on the body manifold B (II.1). The material connection, which we introduce for relativity in (I.19) is derived as a direct extension of a given material connection on B. Since B is paracompact [108, pp. 66-67] material connections exist on B. This is true even of the more general body manifold used in Relativity (I.1), since a differential manifold, by Dieudonné's definition is metrizable [22]. The material connection, which is of course not necessarily unique, provides a method for defining material isomorphisms in a smooth way between B_X and B_Y for all X and Y contained in some smooth curve C. If we construct a triad of vectors spanning B_X i.e. a basis, and transport it to 3 vector fields on $C \subset B$ forming a basis at B_Y for all $Y \in C$, then the response properties of the infinitesimal "material element" at each $Y \in C$ with respect to this frame is the same. That is, if we map this basis to the canonical basis in $\mathbb{R}^3$ and call that reference placement $r(Y)$ then $\mathcal{G}_{r(Y)}$ is independent of Y for $Y \in C$. The material connection and associated material derivative represent the logical way to describe how tensors on B vary relative to the material consistency. A number of propositions about material connections and the associated torsion and curvature tensors for specific types of bodies, especially solids are given by Wang [108].

(P.9) Noll's New Theory of Simple Materials.

In 1972 a new approach was formulated by Noll [77] for expressing the stress in terms of the deformation that involved processes rather than histories (P.3). In this case we no longer need to know or examine the infinitesimal deformation history which requires knowledge of the infinite past. The new theory is more flexible, allowing the introduction of thermodynamic phenomena [77, p. 48] in a relatively easy way. Noll has developed a theorem which relates the original theory of continua to the new theory [77, p. 32] showing that the previous theory covers exactly those materials which are semi-elastic in the new sense. The stress is considered to be an intrinsic second order symmetric tensor function on the body manifold (as a function of time) thus eliminating the need for considering the principle of material frame indifference. As well, because we have a simple material, we can work with B_X the tangent space to the body manifold B at X, rather than considering all of B. We write $\mathcal{T} = B_X$ and look upon $\mathcal{T}$ as an abstract n-dimensional vector space $(n = 3$ in applications) with a special additional structure that we refer to as the structure of a material element [77, p. 12]. We say the material element $\mathcal{T}$ is in state σ if we have selected a particular element σ out of a state space Σ which is associated with $\mathcal{T}$. A configuration or deformation G is determined by σ, and the stress S also is determined, where $G \in \mathrm{Sym}^+(\mathcal{T},\mathcal{T}^*)$ and $S \in \mathrm{Sym}(\mathcal{T}^*,\mathcal{T})$. Any deformation process $P\colon [0, d_p] \to \mathrm{Sym}^+(\mathcal{T},\mathcal{T}^*)$ with initial deformation $P(0) = G$ can be applied to the material element $\mathcal{T}$ in state σ to transform the material element at time d_p to a new state σ'. The new state determines for us the configuration $G' = P(d_p)$ and the final stress

S'. To see this more clearly, we recount here Noll's definition of a material element [77, p. 9, 13].

Configuration space $\mathcal{G}$ is a closed and connected subset of $\mathrm{Sym}^+(\mathcal{T},\mathcal{T}^*)$. If $P\colon [0,d_p] \to \mathcal{G}$ is a process, we call $P^i = P(0)$, $P^f = P(d_p)$ the *initial and final values* of the process. If $G \in \mathcal{G}$, $t \in \mathbb{R}$, $t \geq 0$ we define a process $G_{(t)}\colon [0,t] \to \mathcal{G}$ by $G_{(t)}(r) = G$ $\forall\, r \in [0,t]$, and call it the *freeze of duration* t *at* G. Let P be a process, $t_1,t_2 \in [0,d_p]$, $t_1 \leq t_2$. We define a new process $P_{[t_1,t_2]}$ of duration $t_2 - t_1$ by $P_{[t_1,t_2]}(t) = P(t+t_1)$ for $t \in [0,\ t_2-t_1]$ and call it a *segment* of the process P. Let P_1 and P_2 be processes with $P_1^f = P_2^i$. We define a new process $P_1 * P_2$ of duration $d_{P_1} + d_{P_2}$ called the *continuation of* P_1 *with* P_2 by

$$(P_1 * P_2)(t) = \begin{cases} P_1(t) & \text{if} \quad t \in [0,\ d_{P_1}], \\ P_2(t - d_{P_1}) & \text{if} \quad t \in [d_{P_1},\ d_{P_1} + d_{P_2}]. \end{cases}$$

We say that $(\mathcal{T}, \mathcal{G}, \Pi)$ is a *body element* if Π is a class of processes with values in $\mathcal{G}$ satisfying

1. Any freeze at any $G \in \mathcal{G}$ belongs to Π.
2. If $P \in \Pi$ then every segment of P belongs to Π.
3. Π is closed under continuation, i.e. if $P_1,P_2 \in \Pi$ with $P_1^f = P_2^i$ then $P_1 * P_2 \in \Pi$.
4. If $G_1,G_2 \in \mathcal{G}$, $\exists\, P \in \Pi$ with $P^i = G_1$, $P^f = G_2$.

A *material element* is a septuple $(\mathcal{T}, \mathcal{G}, \Pi, \Sigma, \hat{G}, \hat{S}, \hat{\rho})$ in which $(\mathcal{T}, \mathcal{G}, \Pi)$ is the underlying body element, Σ is a topological space called the *state space* of the material element, $\hat{G}\colon \Sigma \to \mathcal{G}$ is a continuous mapping called the *configuration map*, $\hat{S}\colon \Sigma \to \mathcal{S} = \mathrm{Sym}(\mathcal{T}^*,\mathcal{T})$ is a continuous map called the *stress map*, and $\hat{\rho}\colon (\Sigma \times \Pi)_{\mathrm{fit}} \to \Sigma$ is

called the *evolution map* where $(\Sigma \times \Pi)_{\text{fit}} = \{(\sigma,P) \mid \sigma \in \Sigma,\ P \in \Pi,\ P^i = \hat{G}(\sigma)\}$. The material element is subject to six axioms I through VI which Noll lays out, the first three of which we will consider here. The last three are related to the topology on the state space which he defines in terms of the natural uniformity on $\Sigma_G = \hat{G}^{-1}(G) \subset \Sigma$ for each $G \in \mathcal{G}$, and uses to prove some propositions on relaxed states, accessibility, and semi-elastic materials. Since the formalism we will be using in Chapter IV for thermodynamic material elements uses a different topology on Σ and a different evolution map and some variations from Noll's approach, we will not include Axioms IV, V and VI here.

<u>Axiom I</u> *For all* $(\sigma,P) \in (\Sigma \times \Pi)_{\text{fit}}$, $\hat{G}(\hat{\rho}(\sigma,P)) = P^f$,

<u>Axiom II</u> *If* $\sigma \in \Sigma$, $P_1,P_2 \in \Pi$, $\hat{G}(\sigma) = P_1^i$, $P_1^f = P_2^i$ *then* $\hat{\rho}(\sigma,\ P_1 * P_2) = \hat{\rho}(\hat{\rho}(\sigma,\ P_1),\ P_2)$.

<u>Axiom III</u> *If* $\sigma_1,\sigma_2 \in \Sigma$, $\hat{G}(\sigma_1) = \hat{G}(\sigma_2) = G$ *and* $\hat{S} \circ \rho(\sigma_1,P) = \hat{S} \circ \rho(\sigma_2,P)$ *for all* $P \in \Pi$ *with* $p^i = G$ *then* $\sigma_1 = \sigma_2$.

CHAPTER I

KINEMATICS IN SPACE TIME

(I.1) Introductory Definitions.

We begin by introducing the basic structure of a body manifold and a space-time in relativity, and the foundations for describing its motion. Following Carter and Quintana [13] we begin with a differential manifold M of dimension $n+1$ called the *space-time* and a manifold B of dimension n called the *body manifold*. In all examples used, we will take $n=3$, which fits with our physical experience. The space-time will be equipped with a Lorentz or Galilei structure, Künzle [46] which will be elaborated in greater detail as we proceed. For the purposes of the present chapter it suffices to say that a Lorentz metric tensor exists on M, which we write as g_{ab} in local coordinates, with the usual properties familiar to those who study general relativity. In addition, it is assumed a fibration map (Dieudonné [22] p. 77) namely $P: M \to B$ exists with fibres diffeomorphic to $\mathbb{R}$, and tangent vectors to the fibres time-like everywhere. Thus M is that portion of space-time through which the material medium moves, and we are not considering here those regions of vacuum or empty space. For $X \in B$, the fibres $P^{-1}(X)$ are called *world lines*. We assume that the notion of "future pointing" and "past pointing" are well defined concepts that partition the time like vectors at each point $x \in M$ (considered as vectors in the tangent space M_x) into two distinct classes. We shall see later how this relates to the concept of the Lorentz group structure on M being orthochronous, and the manifold M time-sense preserving. It is

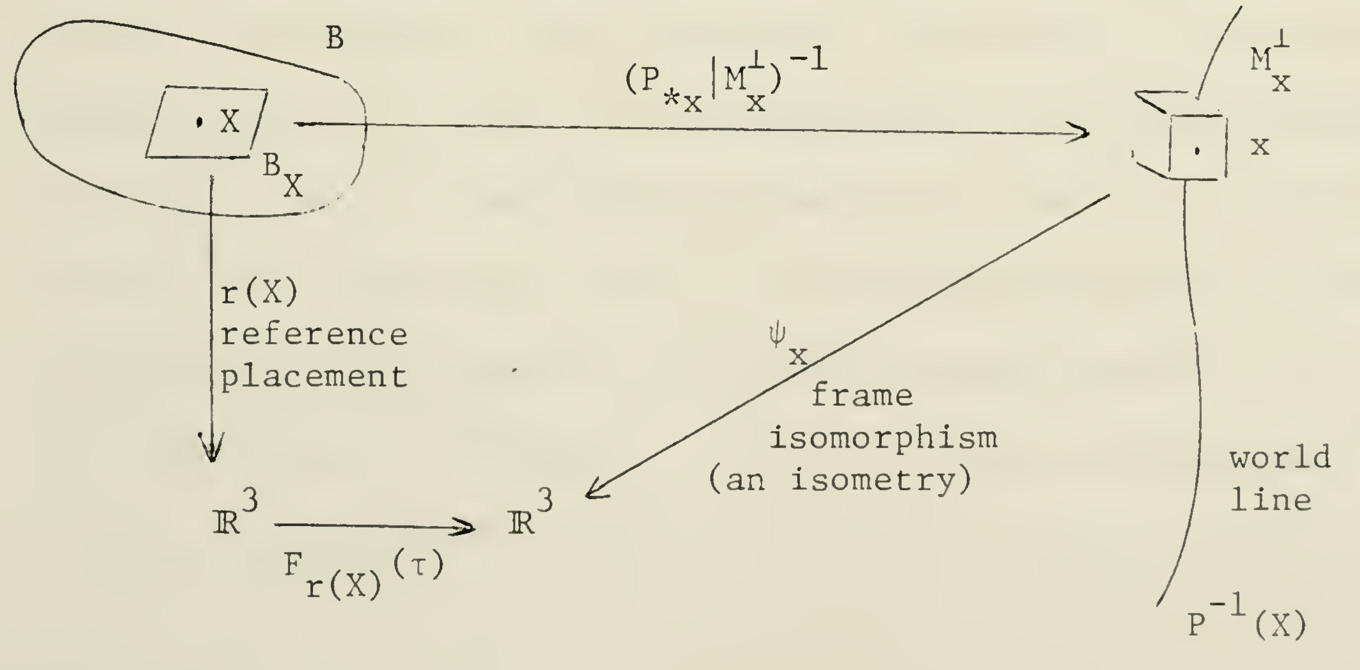
worth noting that the world lines $P^{-1}(X)$ can be parametrized by proper time τ such that the tangent vectors $\frac{dx}{d\tau}$ are future pointing and have length -1. (We assume that $g_{ab}v^a v^b > 0$ for any space-like vectors $\underset{\sim}{v}$.)

The Body manifold B is assumed to be orientable [22, p. 150], so consequently there exists a saturated oriented atlas $\{(U_\alpha, \phi_\alpha), \alpha \in A\}$ of B with U_α an open subset of B and $\phi_\alpha: U_\alpha \to \phi_\alpha(U_\alpha) \subset \mathbb{R}^n$ a diffeomorphism, the charts compatible with positive Jacobian determinants on the overlap, and the family maximal. In fact, there are two such atlases corresponding to the two orientations of B. Likewise B has an everywhere non-zero n-form (completely skew n-covariant tensor field). We assume a particular such n-form is distinguished as a basic property of the body manifold, and denote this form by η calling it the *mass n-form*. If $\alpha \in A$ we call the map $\phi_\alpha: U_\alpha \to \mathbb{R}^n$ a *local placement* of B in $\mathbb{R}^n$, and for $X \in U_\alpha$ we say the derivative map $\phi_{\alpha*X}: B_X \to \mathbb{R}^n$ is an *infinitesimal placement* (or simply a *placement*). Similarly we have the dual linear map $\phi^*_{\alpha X}: \mathbb{R}^3 \to B^*_X$ of the isomorphism $\phi_{\alpha*X}$ where the euclidean norm on $\mathbb{R}^3$ is used to identify $\mathbb{R}^3$ with $\mathbb{R}^{3*}$. The map $\phi^*_{\alpha X} \circ \phi_{\alpha*X}: B_X \to B^*_X$ can be viewed as a positive definite symmetric bilinear form on B_X, the set of which is denoted by $\mathrm{Sym}^+(B_X, B^*_X)$. We call $\phi^*_{\alpha X} \circ \phi_{\alpha*X}$ the *configuration* of X in the placement $\phi_{\alpha*X}$, $X \in U_\alpha$. The configuration or *deformation* determines for us the value of a Riemannian metric at the single point $X \in B$. In more general theories later, which include thermodynamics and even electromagnetism, we will reserve the term deformation for the above, and use configuration in a more general setting. These terms, and the notation above, parallels that of Noll [77]. The notion of an (infinitesimal) reference configuration used by Wang [108] in this system, would be a map of the

type $\phi_{\alpha * X}$. Thus the ϕ_α can be used to describe coordinates of B, physical placements or reference placements. Let us now see how to extend this to relativity.

(I.2) The Deformation Gradient.

The world line $P^{-1}(X)$ can be parametrized by proper time τ. There exists an open interval I in $\mathbb{R}$ (bounded or unbounded) with a function $\beta: I \to P^{-1}(X)$, $x = \beta(\tau) \in P^{-1}(X)$ called the proper time parametrization which is a diffeomorphism. There is a future-pointing time like vector field on M whose value in local coordinates is denoted by u^a called the *flow field* of the material medium. It satisfies $g_{ab}u^a u^b = -1$ everywhere, and $u^a = \frac{dx^a}{d\tau}$ where τ is proper time along world lines. Thus $\underset{\sim}{u}|_x = \left.\frac{d\beta}{d\tau}\right|_{x \in P^{-1}(X)}$. At each $x \in M$, M_x is the tangent space to the differential manifold M at x, and we can define $M_x^\perp$ to be the set of all vectors in M_x orthogonal to $\underset{\sim}{u}|_x$ using the Lorentz metric. The map $P_{*x}: M_x \to B_X$ which is linear (the tangent map to the projection) is an isomorphism when restricted to $M_x^\perp$, i.e. $(P_{*x}|M_x^\perp): M_x^\perp \to B_X$ is bijective. Using this we define the deformation gradient, analogous to the classical case of Truesdell [105] p. 11.



We will do this for the case $n = 3$ although the extension in general is clear. For a fixed $X \subset B$ and reference placement $r(X)\colon B_X \to \mathbb{R}^3$ (a linear isomorphism) and a *frame isomorphism* $\psi_x\colon M_x \to \mathbb{R}^3$ a smooth function defined for all $x \in P^{-1}(X)$ (that is a submanifold of M) we have a *deformation gradient* $F_{r(X)}(\tau)$ defined by $F_{r(X)}(\tau) = \psi_x \circ P_{*x}^{-1} \circ r^{-1}(X)$. We have abbreviated $(P_{*x}|M_x^{\perp})^{-1}$ by $(P_{*x})^{-1}$ without ambiguity. Here $x = \beta(\tau)$ and τ is proper time along the world line $P^{-1}(X)$. The frame isomorphism is assumed to be metric preserving, between the natural Euclidean norm on $\mathbb{R}^3$ and the norm on $M_x^{\perp}$ determined by the tensor whose components in local coordinates are $\gamma_{ab} = g_{ab} + u_a u_b$ evaluated at $x(u_a = g_{ab}u^b)$. Thus if $\underset{\sim}{e}_1 = (1,0,0)^T$, $\underset{\sim}{e}_2 = (0,1,0)^T$, $\underset{\sim}{e}_3 = (0,0,1)^T$ are the standard basis of $\mathbb{R}^3$ we can let $\underset{\sim}{r}|_x = \psi_x^{-1}(\underset{\sim}{e}_1)$, $\underset{\sim}{s}|_x = \psi_x^{-1}(\underset{\sim}{e}_2)$, $\underset{\sim}{t}|_x = \psi_x^{-1}(\underset{\sim}{e}_3)$ for each $x \in P^{-1}(X)$ and obtain smooth vector fields $\underset{\sim}{r}$, $\underset{\sim}{s}$, $\underset{\sim}{t}$ defined on the submanifold $P^{-1}(X)$ of M. For each $x \in P^{-1}(X)$, $\underset{\sim}{r}|_x$, $\underset{\sim}{s}|_x$, $\underset{\sim}{t}|_x$ form an orthonormal basis for $M_x^{\perp}$.

(I.3) Mass Measure.

In the sense of Dieudonné [22] p. 152 we give the Body manifold B the orientation of the oriented atlas that we choose to make the mass n-form η a member of that orientation. Then [22, p. 171] there is a Lebesgue measure m on B corresponding to η for which we may write $m(A) = \int_A \eta$ for A a measurable subset of B. We call m the *mass measure* on B. This mass measure is absolutely continuous in each local configuration with respect to the volume Lebesgue measure dV in $\mathbb{R}^3$ with Radon-Nikodym derivative ρ called the mass density, i.e. for $A \subset U_\alpha$, A measurable, $m(A) = \int_{\phi_\alpha(A)} \rho_\alpha \, dV$. On $U_\alpha \cap U_\beta$ we have

$$\rho_\alpha = J_{\alpha\beta}\rho_\beta \quad \text{where} \quad J_{\alpha\beta} = \det\left(\frac{\partial X^i_\beta}{\partial X^j_\alpha}\right) \quad i,j = 1,2,3, \quad \underset{\sim}{X}_\alpha = (\phi_\alpha \circ \phi_\beta^{-1})(\underset{\sim}{X}_\beta)$$

for $\underset{\sim}{X}_\beta \in \phi_\beta(U_\alpha \cap U_\beta) \subset \mathbb{R}^3$, i.e. $\underset{\sim}{X}_\alpha = \phi_\alpha(X)$, $\underset{\sim}{X}_\beta = \phi_\beta(X)$, $X \in B$. ρ_α is a real valued function on U_α which is smooth and is called the *density* in the α-local configuration, and $J_{\alpha\beta}$ is the *Jacobian*. We can use the charts in our oriented atlas for B indexed by $\alpha \in A$ as coordinates for identifying tensors in component form, as well as for specific physical configurations or deformations. In this sense we can write our mass n-form or mass volume element (for $n = 3$) as $\underset{\sim}{\eta} = \eta_{ijk} dX^i \otimes dX^j \otimes dX^k = \eta_{ijk} dX^i \wedge dX^j \wedge dX^k$, or in specific α or β representations, $\underset{\sim}{\eta} = \overset{\alpha}{\eta}_{ijk} dX^i_\alpha \wedge dX^j_\alpha \wedge dX^k_\alpha = \overset{\beta}{\eta}_{ijk} dX^i_\beta \wedge dX^j_\beta \wedge dX^k_\beta$ where the Einstein summation convention applies on $i,j,k = 1,2,3$. We have $\eta_{ijk} = \epsilon_{ijk}\rho$ (or in α coordinate system $\overset{\alpha}{\eta}_{ijk} = \epsilon_{ijk}\rho_\alpha$) where $\epsilon_{123} = 1$ and ϵ_{ijk} is completely skew. Here α plays a dual role of physical deformation and coordinate. In particular $\eta_{123} = \rho$ and $\underset{\sim}{\eta} = 3!\eta_{123} dX^1 \wedge dX^2 \wedge dX^3$.

(I.4) Motion and Stress.

A *placement process* or motion is a time parametrized family of placements defined for $t \in [0,d_p]$. This can be defined for local or infinitesimal processes with U_α or X respectively fixed throughout the process. If our domain for t is $(-\infty,t_0)$ we have a placement history. In a similar way we can construct deformation or configuration processes and histories which are necessarily infinitesimal. We can from this develop the classical continuum mechanics of Truesdell [105] and others using histories or the New theory of continuum mechanics for simple materials along the lines of Noll [77] using processes. We shall examine the former here briefly in this chapter, and the latter

later on in the more general context of thermodynamics.

The Stress Tensor (Truesdell [105] p. 29) represents the contact forces across a smooth surface in a body B, and is the starting point for the dynamical description. If $A \subset U_\alpha \subset B$ is a volume bounded by a surface ∂A, then the force on A can be written as

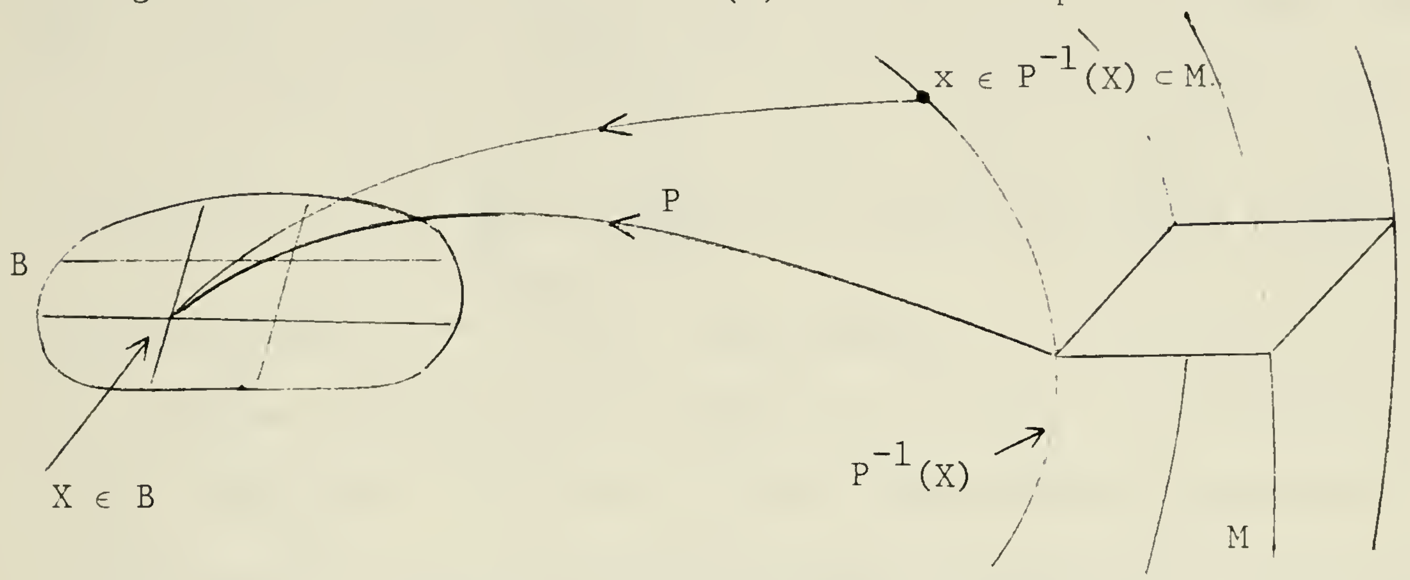
$$\underset{\sim}{f}(A) = \underbrace{\underset{\sim}{f}_b(A)}_{\substack{\text{External}\\ \text{Body Force}}} + \underbrace{\underset{\sim}{f}_c(A)}_{\substack{\text{Contact}\\ \text{Force}}} \quad \text{where} \quad \underset{\sim}{f}_b(A) = \int_{\phi_\alpha(A)} \underset{\sim}{b}\,\rho_\alpha\, dV = \int_A \underset{\sim}{b}\, dm$$

and $\underset{\sim}{f}_c(A) = \int_{\partial\phi_\alpha(A)} \underset{\sim}{t}\, dS$. [105, p. 26]. We say $\underset{\sim}{b}$ is the specific body force and $\underset{\sim}{t}$ is the traction, which is given by $\underset{\sim}{t} = \underset{\sim}{T}\,\underset{\sim}{n}$ for simple tractions ($\underset{\sim}{n}$ is normal to the surface $\partial\phi_\alpha(A)$, and T is the Stress tensor). The Stress T satisfies Cauchy's two laws of continuum mechanics [105, p. 31] $\rho\ddot{\underset{\sim}{x}} = \operatorname{div} T + \rho\underset{\sim}{b}$ and $T^T = T$, (P.4) if traction is simple and all torques are moments of forces, i.e. we have nothing like quantum mechanical spin.

Of course in relativity, especially general relativity we are not able to work with local placements and even infinitesimal placements present problems, so we are left with configurations and deformations. This appears to limit us to a class of materials which is referred to in the literature as simple materials (P.5), and also requires us to investigate the principle of material frame indifference, see [105, pp. 33-36], Wang [108] p. 39. In order to discuss the principle for determinism of the stress for simple materials, and frame indifference, let us examine frame isomorphisms and vector transport theory. For further discussion on material frame indifference you should see Maugin [66,67], Bressan [8], and for Fermi transport, Lianis [50] pp. 62-64, Enosh and Kovetz [27] and Krause [45].

(I.5) Vector Transport in Relativity.

We seek a method to parallel transport a vector at $x \in P^{-1}(X)$ along the entire world line $P^{-1}(X)$. This transport must have the



property of preserving orthogonal vectors, i.e. a vector in $M_x^{\perp}$ must be transported to a vector in $M_{x'}^{\perp}$, for $x, x' \in P^{-1}(X)$, and it must preserve orthonormality of any basis of $M_x^{\perp}$.

For any vector field $\underset{\sim}{v}$ defined on $P^{-1}(X)$ with local coordinates v^a, ($a = 1,2,3,4$ — metric g_{ab} has signature type (1,1,1,-1)) we define the *parametric covariant derivatives* $\frac{\delta v^a}{d\tau} = \frac{dv^a}{d\tau} + v^i \Gamma_{i\,j}^{\ a} \frac{dx^j}{d\tau}$ and $\frac{\delta v_a}{d\tau} = \frac{dv_a}{d\tau} - v_i \Gamma_{a\,j}^{\ i} \frac{dx^j}{d\tau}$ with respect to the connection Γ. Following Hehl [39,40] we write $\Gamma_{i\,j}^{\ a} = \left\{ {a \atop i\,j} \right\} - K_{i\,j}^{\ a}$ and call $K_{i\,j}^{\ a}$ the contorsion tensor. It is worth noting at this point that the connection $\Gamma_{i\,j}^{\ a}$, which is not symmetric in general like the Christoffel symbols $\left\{ {a \atop i\,j} \right\}$, is not used to describe the dynamics of space-time like in Hehl's theory and in Einstein-Cartan theory with the presence of (quantum mechanical) spin, but the rotation of the material medium. Dynamically we assume the usual Einstein equations hold unless otherwise specifically stated. In order that $g_{ab|c} = 0$ we require that $K_{iaj} = K_{[ia]j}$ where the index on the contorsion tensor is lowered using the metric. We let $|$ denote Γ covariant differentiation and ; denote Christoffel symbol covariant differentiation.

If r^a are the components of an orthogonal transport invariant vector field defined on $P^{-1}(X)$ we may write the transport equation as $\frac{\delta r^a}{d\tau} = c(\tau)u^a + d(\tau)v^a$ where $\underset{\sim}{v}(x) \in M_x^{\perp}$ for all $x \in P^{-1}(X)$. Then $0 = \frac{\delta}{d\tau}(u_a r^a) \Rightarrow c(\tau) = r^b \frac{\delta u_b}{d\tau}$. If $\underset{\sim}{s}$ is also transport invariant and orthogonal we have $\frac{\delta r^a}{d\tau} = u^a \left(\frac{\delta u_b}{d\tau}\right) r^b + d(\tau)v^a$ and $\frac{\delta s_a}{d\tau} = u_a \left(\frac{\delta u^b}{d\tau}\right) s_b + \hat{d}(\tau)\hat{v}_a$. Then we require $0 = \frac{\delta(r^a s_a)}{d\tau}$ which implies $0 = \hat{d}(\tau)r^a\hat{v}_a + d(\tau)s_a v^a$. The simplest solution made by taking $\hat{v}_a = v^a = 0$ is called the *Fermi-Walker transport*. It is the only such transport which preserves orthonormality of *all* orthonormal frames irrespective of the choice of the connection.

If $\underset{\sim}{r}, \underset{\sim}{s}, \underset{\sim}{t}$ is an oriented orthonormal frame at $x \in P^{-1}(X)$ (a basis for $M_x^{\perp}$ oriented by image under P_{*x} in B_X) then we can define an oriented frame of orthonormal vector fields on $P^{-1}(X)$ by the Fermi transport condition, namely $\frac{\delta r^a}{d\tau} = u^a\left(\frac{\delta u_b}{d\tau}\right)r^b$, $\frac{\delta s^a}{d\tau} = u^a\left(\frac{\delta u_b}{d\tau}\right)s^b$, $\frac{\delta t^a}{d\tau} = u^a\left(\frac{\delta u_b}{d\tau}\right)t^b$. Now let δ and $|$ denote covariant differentiation with respect to Γ and $\bar{\delta}$ and $||$ covariant differentiation w.r.t. $\bar{\Gamma}$. Let $(\underset{\sim}{r},\underset{\sim}{s},\underset{\sim}{t})$ be Fermi transported along $P^{-1}(X)$ using Γ and $(\bar{\underset{\sim}{r}},\bar{\underset{\sim}{s}},\bar{\underset{\sim}{t}})$ using $\bar{\Gamma}$. We write $\Gamma_{i\ j}^{\ a} = \left\{{a \atop i\ j}\right\} - K_{i\ j}^{\ a}$ and $\bar{\Gamma}_{i\ j}^{\ a} = \left\{{a \atop i\ j}\right\} - \bar{K}_{i\ j}^{\ a}$ where the contorsion tensors satisfy $K_{iaj} = K_{[ia]j}$ and $\bar{K}_{iaj} = \bar{K}_{[ia]j}$ in order that $g_{ab|c} = 0 = g_{ab||c}$. Such contorsions are called *metric* or *Lorentz*, and the same adjectives apply to the corresponding connections.

The two bases of orthonormal vector fields on $P^{-1}(X)$ can be related by an orthogonal matrix function of proper time. Writing $\underset{\sim}{r}_{(1)} = \underset{\sim}{r}$, $r_{(2)} = \underset{\sim}{s}$, $r_{(3)} = \underset{\sim}{t}$ we have $\bar{\underset{\sim}{r}}_{(i)}(\tau) = Q_i^{\ j}(\tau)\underset{\sim}{r}_{(j)}(\tau)$, and hence $Q_i^{\ j} = \bar{\underset{\sim}{r}}_{(i)} \cdot \underset{\sim}{r}_{(j)}$ $i,j = 1,2,3$. The indices in parentheses have no tensor significance and can be raised or lowered freely without any change in

the vectors and without using g_{ab}, i.e. $\underset{\sim}{r}_{(j)} = \underset{\sim}{r}^{(j)}$, $Q_i{}^j = Q_{ij} = Q^i{}_j$. Clearly $Q_i{}^j Q_{kj} = \delta_{ik}$ $i,k = 1,2,3$, with sum on j. Introducing a free tensor index to represent components in a coordinate system, we write the Fermi transport conditions as

$$\frac{\delta r^a_{(i)}}{d\tau} = u^a \left(\frac{\delta u_b}{d\tau}\right) r^b_{(i)}, \quad \frac{\bar{\delta}\bar{r}^a_{(i)}}{d\tau} = u^a \left(\frac{\bar{\delta} u_b}{d\tau}\right) \bar{r}^b_{(i)} \quad \text{for } i = 1,2,3.$$

Now $Q_{ij} = \bar{r}^{(i)}_a r^a_{(j)}$ so that we have,

$$\frac{dQ_{ij}}{d\tau} = \bar{r}^{(i)}_a \frac{dr^a_{(j)}}{d\tau} + r^a_{(j)} \frac{d\bar{r}^{(i)}_a}{d\tau} = \bar{r}^{(i)}_a \left(\frac{\delta r^a_{(j)}}{d\tau} - r^b_{(j)} \Gamma^a_{b\,c} \frac{dx^c}{d\tau}\right)$$
$$+ r^a_{(j)} \left(\frac{\bar{\delta}\bar{r}^{(i)}_a}{d\tau} + \bar{r}^{(i)}_b \bar{\Gamma}^b_{a\,c} \frac{dx^c}{d\tau}\right).$$

Using the Fermi transport relations and the fact that $\underset{\sim}{u}$ is orthogonal to $\underset{\sim}{r}_{(j)}$ and $\bar{\underset{\sim}{r}}_{(j)}$ for each j we have

$$\frac{dQ_{ij}}{d\tau} = (\bar{\Gamma}^a_{b\,c} - \Gamma^a_{b\,c})\bar{r}^{(i)}_a r^b_{(j)} u^c = (K_{bac} - \bar{K}_{bac})\bar{r}^a_{(i)} r^b_{(j)} u^c$$

using $u^c = \frac{dx^c}{d\tau}$. Thus $\frac{dQ_{ij}}{d\tau} = (\bar{K}_{abc} - K_{abc})\bar{r}^a_{(i)} r^b_{(j)} u^c$, and putting $\Delta K_{abc} = \bar{K}_{abc} - K_{abc}$, using $\bar{r}^a_{(i)} = Q_{ik} r^a_{(k)}$ we get

$\frac{dQ_{ij}}{d\tau} = \Delta K_{abc} Q_{ik} r^a_{(k)} r^b_{(j)} u^c = Q_{ik}\Lambda_{kj}$ with summation on k. Here $\Lambda_{kj} = \Delta K_{abc} r^a_{(k)} r^b_{(j)} u^c = -\Lambda_{jk}$ so $\Lambda_{jk} = \Lambda_{[jk]}$. The condition

$\frac{dQ_{ij}}{d\tau} = Q_{ik}\Lambda_{kj}$ guarantees Q remains orthogonal for all τ since Λ is antisymmetric. $[\frac{dQ}{d\tau} = Q\Lambda \Rightarrow \frac{d}{d\tau}(Q^T Q) = 0$ since $\Lambda + \Lambda^T = 0$.]

(I.6) Material Frame Indifference and Frame Indifferent Tensors.

The introduction of rotating frames permits us to understand the principle of material frame indifference in relativity for a simply body. We can define our frame identification maps $\psi_x : M^\perp_x \to \mathbb{R}^3$ for each

$x \in P^{-1}(X)$ by $\psi_x(\underset{\sim}{r}_{(i)}(x)) = \underset{\sim}{e}_{(i)}$, $\bar{\psi}_x(\bar{\underset{\sim}{r}}_{(i)}(x)) = \underset{\sim}{e}_{(i)}$ for $i = 1,2,3$ and extend by linearity. These linear isomorphisms preserve norm and inner product, since the canonical basis of $\mathbb{R}^3$ is also orthonormal. The infinitesimal motion of X is given by $\kappa_\tau = \psi_x \circ P_{*x}^{-1}\colon B_X \to \mathbb{R}^3$ where $x = \beta(\tau)$ and where $P_{*x}\colon M_x^{\perp} \to B_X$ is the fundamental isomorphism induced by P. The stress tensor is then given by the constitutive equation (P.5) $T(x) = \underset{s=-\infty}{\overset{\tau}{F}}(\kappa_s, X)$ where $\tau = \beta^{-1}(x)$, $x \in P^{-1}(X)$, using the principle for the determinism of the stress for simple materials and following the notation of Wang [108] p. 39. T is a symmetric and second order tensor on $\mathbb{R}^3$, and in the notation of Noll [77], $T \in \mathrm{Sym}(\mathbb{R}^3) = \mathrm{Sym}(\mathbb{R}^3, \mathbb{R}^3)$ (P.2). $T(x)$ is clearly independent of the choice of proper time parametrization β according as where $\beta(0)$ is taken, however it does depend on the choice of the isomorphisms ψ_x (i.e. on the rotation frame). We call F the *response functional* for the simple material.

We can use the ψ_x to transform the $\mathbb{R}^3$ tensor $T(x)$ to a tensor $\hat{T}(x)$ at $x \in M$ on M itself. If we require that $\hat{T}(x)$ is to be independent of the choice of the rotating frame, then automatically F satisfies the principle of material frame indifference and conversely. We can see this as follows. We have $\bar{\underset{\sim}{r}}_{(i)}(\tau) = Q_i{}^j \underset{\sim}{r}_{(j)}(\tau)$ and so for $x = \beta(\tau)$,

$$\bar{\hat{T}} = \bar{T}^{ij} \bar{\underset{\sim}{r}}_{(i)} \otimes \bar{\underset{\sim}{r}}_{(j)} = \Big(\underset{s=-\infty}{\overset{\tau}{F}}(\bar{\kappa}_s, X)\Big)^{ij} Q_i{}^k(\tau) Q_j{}^\ell(\tau) \underset{\sim}{r}_{(k)} \otimes \underset{\sim}{r}_{(\ell)}(\tau)$$

$$= \Big(\underset{s=-\infty}{\overset{\tau}{F}}(\bar{\psi}_{\beta(s)} \circ P_{*\beta(s)}^{-1}, X)\Big)^{ij} Q_i{}^k Q_j{}^\ell \underset{\sim}{r}_{(k)} \otimes \underset{\sim}{r}_{(\ell)}(\tau).$$

Notice that $\psi_x(\underset{\sim}{r}_{(i)}) = \underset{\sim}{e}_{(i)} = \bar{\psi}_x(\bar{\underset{\sim}{r}}_{(i)})$, and so $\bar{\psi}_x(Q_i{}^k \underset{\sim}{r}_{(k)}) = \psi_x(\underset{\sim}{r}_{(i)}) = Q_i{}^k \bar{\psi}_x(\underset{\sim}{r}_{(k)}) \Rightarrow \bar{\psi}_x(\underset{\sim}{r}_{(k)}) = (Q^T)_k{}^i \psi_x(\underset{\sim}{r}_{(i)}) = Q^i{}_k \psi_x(\underset{\sim}{r}_{(i)})$. Hence

$\bar{\psi}_x \circ \psi_x^{-1}(a^k \underset{\sim}{e}_{(k)}) = \bar{\psi}_x(a^k \underset{\sim}{r}_{(k)}) = a^k \bar{\psi}_x(\underset{\sim}{r}_{(k)}) = a^k Q^i{}_k \underset{\sim}{e}_{(i)}$, so that in matrix form $\bar{\psi}_x \circ \psi_x^{-1} = Q$, or $\bar{\psi}_x = Q \circ \psi_x$. Substituting above we get

$$\bar{\hat{T}}(x) = \left(\underset{s=-\infty}{\overset{\tau}{F}} \left[Q(s) \circ \underbrace{\psi_{\beta(s)} \circ P^{-1}_{*\beta(s)}}_{\kappa_s}, X \right] \right)^{ij} Q_i{}^k(\tau) Q_j{}^\ell(\tau) \underset{\sim}{r}_{(k)} \otimes \underset{\sim}{r}_{(\ell)}(\tau).$$

Also,

$$\hat{T}(x) = \left(\underset{s=-\infty}{\overset{\tau}{F}} (\kappa_s, X) \right)^{k\ell} \underset{\sim}{r}_{(k)} \otimes \underset{\sim}{r}_{(\ell)}(\tau).$$

Hence $\hat{T} = \bar{\hat{T}}$ if and only if

$$\underset{s=-\infty}{\overset{\tau}{F}} \left[Q(s) \circ \kappa_s, X \right] = Q(\tau) \left[\underset{s=-\infty}{\overset{\tau}{F}} (\kappa_s, X) \right] Q(\tau)^T, \qquad \text{[c.f. (P.7)].}$$

This must hold for all orthogonal tensor histories $Q(\tau)$ in order for the principle of material frame indifference to hold as well as for the stress tensor $\hat{T}$ to be a well defined orthogonal space time tensor field on $P^{-1}(X)$ and hence on M. This result is a relativistic generalization of Noll's statement that the use of the intrinsic stress on the body manifold or body element eliminates the need to specifically consider the principle of frame indifference Noll [77], pp. 2, 12.

Recall that for $X \in B$, $x \in P^{-1}(X)$, and $r(X): B_X \to \mathbb{R}^3$ an (orientation preserving) reference isomorphism we had defined the deformation gradient $F_{r(X)}(\tau) = \psi_x \circ P^{-1}_{*x} \circ r^{-1}(X)$, $x = \beta(\tau)$. We can write $F = RU = VR$ where R is orthogonal and V, U are positive definite symmetric matrices uniquely determined. (We use the canonical basis of $\mathbb{R}^3$ for matrix representations.) By orientability $\det R = +1$, since $\det F > 0$.

Let us see how these transform under a change of frame. As before $\bar{\psi}_x \circ \psi_x^{-1} = Q(\tau)$ so $\bar{F} = QF$ where Q is orthogonal. Hence $\bar{R}\bar{U} = QRU$ so $\bar{R} = QR$ and $\bar{U} = U$ by uniqueness of polar decomposition. Therefore $\bar{V} = \bar{R}\,\bar{U}\,\bar{R}^T = Q\,R\,U(QR)^T = Q\,V\,Q^T$. Hence V is frame indifferent, i.e.

$\bar{V} = Q\,V\,Q^T$ (Truesdell [105], p. 23). The stress tensor on $\mathbb{R}^3$ we saw was frame indifferent and as a consequence could be lifted to a stress tensor on $P^{-1}(X)$. Likewise the left stretch tensor V can be lifted to a left stretch orthogonal space time tensor $\hat{V}$ independent of the choice of frame. This holds on $P^{-1}(X)$. If $U \subset B$ is open and $r\colon X \in U \to r(X)$ is a smooth field we can lift V to a smooth field $\hat{V}$ on $P^{-1}(U) \subset M$. We write $\hat{V}(x) = V^{ij}(\tau)\underset{\sim}{r}_{(i)} \otimes \underset{\sim}{r}_{(j)}$ and $\hat{\bar{V}}(x) = \bar{V}^{ij}(\tau)\bar{\underset{\sim}{r}}_{(i)} \otimes \bar{\underset{\sim}{r}}_{(j)}(\tau)$ where $x = \beta(\tau)$. Then clearly $\hat{V} = \hat{\bar{V}}$ because of this frame indifference. We define the *left Cauchy-Green tensor* B by $B = V^2 = F\,F^T$, and we have the left Cauchy-Green tensor $\hat{B}$ on $P^{-1}(X)$ since B has the same property of being frame indifferent $(\bar{B} = Q\,B\,Q^T)$ as V has. Of course these lifted tensors do depend on the choice of the infinitesimal reference placement $r(X)$. The right Cauchy-Green tensor $C = U^2 = F^T F$ satisfies $\bar{C} = C$ like U and is therefore naturally an $\mathbb{R}^3$ tensor and not a tensor on space time like B. We can then clearly write for the stress (in $\mathbb{R}^3$) of our simple material, $T(x) = \mathop{\mathcal{F}}_{s=-\infty}^{\tau} (\kappa_s, X)$, where $x = \beta(\tau) \in P^{-1}(X)$ and so

$$T(x) = \mathop{\mathcal{F}}_{s=-\infty}^{\tau} (\psi_{\beta(s)} \circ P^{-1}_{*\beta(s)}, X) = \mathop{\mathcal{G}_{r(X)}}_{s=-\infty}^{\tau} \left(\psi_{\beta(s)} \circ P^{-1}_{*\beta(s)} \circ r^{-1}(X), X\right)$$

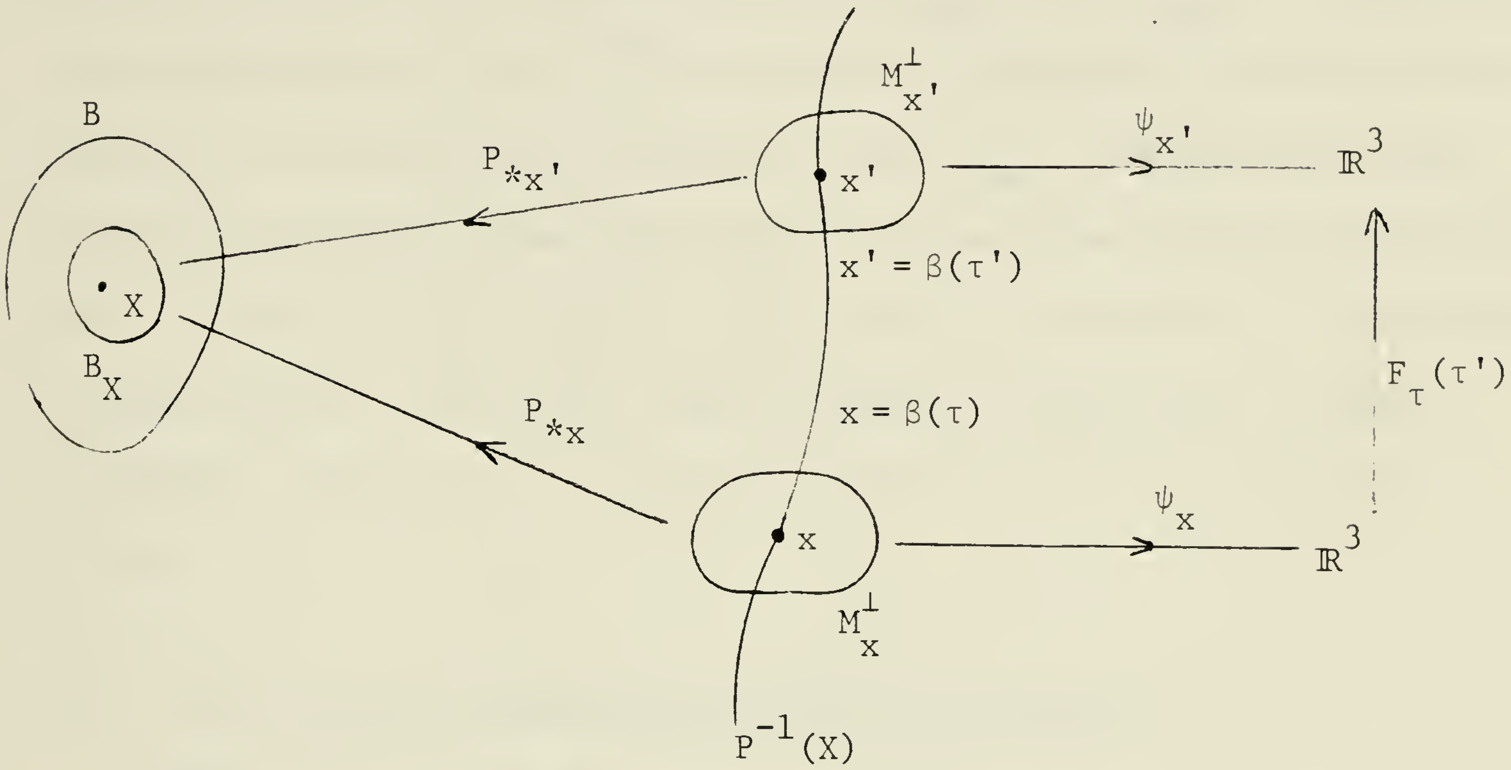
$$= \mathop{\mathcal{G}_{r(X)}}_{s=-\infty}^{\tau} \left(F_{r(X)}(s), X\right)$$

using the notation (P.5) of Wang [108], p. 39.

(I.7) Relative Deformation Gradients and Material Frames.

We can express the deformation of a material medium at a point $x' \in P^{-1}(X)$ in terms of another point $x \in P^{-1}(X)$ by means of the relative deformation gradient $F_\tau(\tau')$ which is defined by

$F_\tau(\tau') = \psi_{x'} \circ P^{-1}_{*x'} \circ P_{*x} \circ \psi_x^{-1}$. Clearly $F_\tau(\tau) = I$ the identity on $\mathbb{R}^3$ and under a change of frame $\psi_x \to \bar{\psi}_x$ we have, $\bar{F}_\tau(\tau') = \bar{\psi}_{x'} \circ P^{-1}_{*x'} \circ P_{*x} \circ \bar{\psi}_x^{-1} = Q(\tau')\psi_{x'} \circ P^{-1}_{*x'} \circ P_{*x} \circ \psi_x^{-1} Q^T(\tau) = Q(\tau')F_\tau(\tau')Q^T(\tau)$. Recall that



$\frac{dQ_{ij}}{d\tau} = \lambda_{kj} Q_i{}^k$ where $\lambda_{ik} = \Delta K_{abc} r^a_{(i)} r^b_{(k)} u^c = \lambda_{[ik]}$. Clearly every smooth orthogonal matrix function $Q(\tau)$ represents a change of frame. We need only select $\Lambda(\tau)$ to satisfy $\Lambda = Q^{-1}\left(\frac{dQ}{d\tau}\right)$ which is automatically anti-symmetric. We then select the contorsion difference to be *orthogonally invariant*, i.e. $\Delta K_{abc} = -B_{ab} u_c$ where $B_{ab} = B_{[ab]}$. Then $\lambda_{kj} = B_{ab} r^a_{(k)} r^b_{(j)}$ uniquely determines B_{ab} if B_{ab} is orthogonal. (We use orthogonal in the sense of Carter and Quintana [13] that any contraction with the flow vector is zero.)

If $r(X)$ is an infinitesimal reference placement of X, we have $F_\tau(\tau') = F_{r(X)}(\tau') \circ F^{-1}_{r(X)}(\tau)$ and under a change of frame

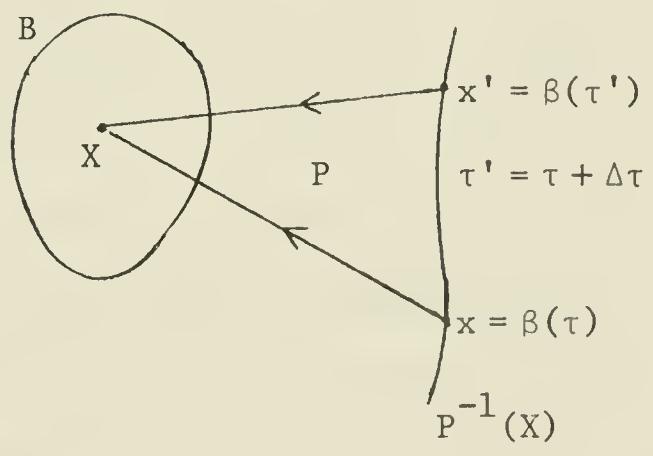
$$\bar{F}_\tau(\tau') = Q(\tau')F_{r(X)}(\tau') \circ F^{-1}_{r(X)}(\tau)Q^T(\tau) = \bar{F}_{r(X)}(\tau') \circ \bar{F}^{-1}_{r(X)}(\tau).$$

For any given $r(X)$ we can choose $Q(\tau)$ for all τ such that $Q(\tau)F_{r(X)}(\tau)$ is positive definite symmetric and equal to, say $U_{r(X)}(\tau)$.

Then we have $\bar{F}_\tau(\tau') = U_{r(X)}(\tau')U^{-1}_{r(X)}(\tau)$ and we say that the bar frame determined by Q is the *material frame* of X relative to the reference $r(X)$. This frame is unique by uniqueness of the polar decomposition Truesdell [105], p. 17. In this frame $\bar{F}_{r(X)}(\tau) = U_{r(X)}(\tau)$. There is a unique connection Γ on $P^{-1}(X)$ which is orthogonally invariant with respect to the Levi-Civita connection with B_{ab} orthogonal and which defines the material frame with respect to the given reference placement $r(X)$. We write $\Gamma_b{}^a{}_c = \left\{{a \atop b\ c}\right\} + B_b{}^a u_c$ for its components in coordinates. In many cases we will have a smooth field of infinitesimal reference placements $r(X)$, $X \subset U$ an open subset of B, so Γ is defined on $P^{-1}(U)$.

(I.8) Lie and Convective Transports and Derivatives.

Define $\phi_{\Delta\tau}\colon M \to M$ by $\phi_{\Delta\tau}(x) = \beta\left(\Delta\tau + \beta^{-1}(x)\right)$ where β is a proper time parametrization of $P^{-1}\left(P(x)\right)$, i.e. $\phi_{\Delta\tau}$ are the parameter difference transports along the integral curves of the vector field u^a. Their derivatives $\phi_{\Delta\tau *}$ give the Lie transports along the integral curves $P^{-1}(X)$ of u^a. We assume for the moment that the interval I on which β is defined is all of $\mathbb{R}$, although we can drop this requirement with slight additional complications on domain and range.



Clearly $\phi_{\Delta\tau_1 + \Delta\tau_2} = \phi_{\Delta\tau_1} \circ \phi_{\Delta\tau_2} = \phi_{\Delta\tau_2} \circ \phi_{\Delta\tau_1}$, ϕ_0 = identity on M, $\phi_{-\Delta\tau} = \phi^{-1}_{\Delta\tau}$, and $P \circ \phi_{\Delta\tau} = P$ on M. Taking derivatives we have on the tangent spaces, $P_{*x'} \circ \phi_{\Delta\tau *} = P_{*x}\colon M_x \to B_X$ (where $P(x) = X$) and on the cotangent spaces, $\phi^*_{\Delta\tau} \circ P^*_{x'} = P^*_x\colon B^*_X \to M^*_x$. The $\phi_{\Delta\tau *}$ and $\phi^*_{\Delta\tau}$ are isomorphisms at each point of M. Clearly P_{*x}

sends $\underset{\sim}{u}$ to zero at each x, and also the contraction of any covector in the image of P_x^* with $\underset{\sim}{u}$ at each x is zero. Hence $\phi_{\Delta\tau}^*$ preserves orthogonality of space-time covectors, and in general any orthogonal covariant tensor at x becomes an orthogonal tensor field on $P^{-1}(X)$ under Lie transport. Equivalently the Lie derivative of a covariant orthogonal tensor field with respect to $\underset{\sim}{u}$ is a covariant orthogonal tensor field of the same rank. Also we see that $\phi_{\Delta\tau *}(\underset{\sim}{u}_x) \in \ker P_{*x}$, and in fact $\phi_{\Delta\tau *}(\underset{\sim}{u}_x) = \underset{\sim}{u}_x$ since $L_{\underset{\sim}{u}}\underset{\sim}{u} = 0$. Hence we have $\mathrm{Im}(P_x^*) = M_x^{*\perp}$ and $P_x^* : B_X^* \to M_x^{*\perp}$ is an isomorphism. It is also easy to see that $(P_{*x}|M_x^\perp) : M_x^\perp \to B_X$ is an isomorphism. These facts we have used before.

Let $r(X)$, $X \in V$ be a smooth family of infinitesimal reference placements on an open set $V \subset B$, $r(X) : B_X \to \mathbb{R}^3$. Then r determines a positive definite Riemannian metric $g_{\alpha\beta}$, $\alpha,\beta = 1,2,3$ on V such that $r(X)$ preserves inner products between vectors, $X \in V$. Thus $g_{\alpha\beta}(X)$ determines an isomorphism between B_X and B_X^* for each $X \in V$ which can be lifted to an isomorphism between $M_x^\perp$ and $M_x^{*\perp}$ for each $x \in P^{-1}(V)$. This can be represented uniquely by an orthogonal space

$$\begin{tikzcd}
B_X^* \arrow[r, "P_x^*"] & M_x^* \\
B_X \arrow[u, "g_{\alpha\beta}(X)"] & M_x^\perp \arrow[l, "P_{*x}|M_x^\perp"'] \arrow[u, "\beta_{ab}(x)"']
\end{tikzcd}$$

time tensor called the *beta tensor* written in local coordinates as β_{ab}, relative to the field $r(X)$ of (infinitesimal) reference placements. Notice $a,b = 1,2,3,4$, $\beta_{ab}u^b = 0$ and $\beta_{ab} = \beta_{ba}$. We say a tensor field on $P^{-1}(X)$ which is orthogonal is *materially invariant* if it maps through P_* to a fixed tensor at X. Clearly β_{ab} maps a materially invariant vector field on $P^{-1}(X)$ to a materially invariant covector field, which is also Lie invariant. The

materially invariant orthogonal vector field need not be Lie invariant however.

Since Lie transport preserves orthogonality of covectors, we can extend it uniquely to a convective transport of arbitrary covectors by demanding that u_a be convectively invariant. To convectively transport v_a we write it as $v_a = w_a + u_a$ (a unique decomposition) where $(w_a) \in M_x^{*\perp}$ at x. We then have by convective transport, $v_a(x') = w_a(x') + \lambda u_a(x')$ where $\left(w_a(x')\right) = P^*_{x'} \circ P_x^{*-1}\left(w_a(x)\right)$ where $\underset{\sim}{w}(x) = \left(w_a(x)\right)$ in component notation. The convective derivative of a covector, i.e. the proper time comparitive derivative at x is given, using the Lie derivative formula (I.10.1) by

$$\mathcal{D}v_a = v_{a;b}u^b + v_c(u^c{}_{;a} + u^c\dot{u}_a)$$

where $\dot{u}_a = u_{a;b}u^b$. This reduces to the Lie derivative if v_a is orthogonal space time i.e. $v_a u^a = 0$, and moreover it satisfies $\mathcal{D}u_a = 0$. Hence $\mathcal{D}v_a = 0$ is the condition for convective transport of covectors. Since it agrees with the Lie derivative on orthogonal covectors we see that $\mathcal{D}$ preserves orthogonal covectors, or equivalently convective transport of an orthogonal covector at x gives an orthogonal covector field on $P^{-1}\left(P(x)\right)$.

We can determine what the convective derivative of a vector field is by the condition $\mathcal{D}(w^a v_a) = \frac{d}{d\tau}(w^a v_a)$ and by the product rule. This forces us to take $\mathcal{D}w^a = w^a{}_{;b}u^b - w^c(u^a{}_{;c} + u^a\dot{u}_c)$. Clearly this preserves orthogonal vectors since $\mathcal{D}(w^a u_a) = 0$, i.e. convective transport of an orthogonal vector at x gives an orthogonal vector field on $P^{-1}\left(P(x)\right)$, and also $w^a u_a = 0$ implies $u_a\mathcal{D}w^a = 0$ so the convective derivative of an orthogonal vector field is an orthogonal vector field.

This can also be proved directly from the component form above.

By the Leibnitz differentiation rule we can extend to get the Convective derivative of an arbitrary tensor field,

$$\begin{aligned}\mathcal{D}T^{ab\cdots}{}_{cd\cdots} &= T^{ab\cdots}{}_{cd\cdots;e}u^e - T^{eb\cdots}{}_{cd\cdots}(u^a{}_{;e} + u^a\dot{u}_e) \\ &- T^{ae\cdots}{}_{cd\cdots}(u^b{}_{;e} + u^b\dot{u}_e) - \cdots + T^{ab\cdots}{}_{ed\cdots}(u^e{}_{;c} + u^e\dot{u}_c) \\ &+ T^{ab\cdots}{}_{ce\cdots}(u^e{}_{;d} + u^e\dot{u}_d) + \cdots .\end{aligned} \qquad \text{(I.8.1)}$$

For additional discussion on the convective derivative see Carter and Quintana [13].

The convective transport of an orthogonal vector along $P^{-1}(X)$ is merely the material transport, i.e. the isomorphism between $M^{\perp}_x$ and $M^{\perp}_{x'}$ is $(P_{*x'}|M^{\perp}_{x'})^{-1} \circ (P_{*x}|M^{\perp}_x) : M^{\perp}_x \to M^{\perp}_{x'}$. Also we see that $\mathcal{D}u^a = 0$ so as in the covariant case convective transport preserves $\underset{\sim}{u}$ and is the natural material isomorphism between the orthogonal tangent spaces. Since materially constant tensor fields are convective transport invariant we see that $\mathcal{D}\beta_{ab} = 0$ for the beta tensor relative to any reference placement field (or even a single infinitesimal reference placement of X if β_{ab} is defined only on $P^{-1}(X)$). Since β_{ab} is orthogonal to the flow u^b, it has an "inverse" α^{ab} which is unique symmetric and orthogonal satisfying $\alpha^{ab}\beta_{bc} = \gamma^a_c = \delta^a_c + u^a u_c$. Then α^{ab} is also materially constant, i.e. $\mathcal{D}\alpha^{ab} = 0$. Clearly $\mathcal{D}\gamma^a_c = 0$ but $\mathcal{D}\gamma_{ab} \neq 0$ in general. (Note: $\gamma_{ab} = g_{ab} + u_a u_b$, $\gamma_{ab}u^b = 0$ and $\gamma^{ab}\gamma_{bc} = \gamma^a_c$). Since β_{ab} is orthogonal and a covariant tensor, $L_u\beta_{ab} = 0$. α^{ab} represents the inverse transformation between $M^{\perp}_x$ and $M^{*\perp}_x$ of the isomorphism determined by $\beta_{ab}(x)$. It is worth noting here that the same $g_{\alpha\beta}$ is induced at X by $r(X)$ and $r'(X)$ if and only if $r'(X) \circ r^{-1}(X) : \mathbb{R}^3 \to \mathbb{R}^3$ is norm preserving (and so represented by an orthogonal matrix).

(I.9) The Induced Metric Function.

At $X \in B$ we can describe a Riemannian metric tensor $g_{\alpha\beta}(\tau)$ at X which is a function of the proper time τ at that point. The transformation is designed so as to make this diagram of isomorphisms commute.

$$\begin{tikzcd}
B^*_X \arrow[r, "P^*_x"] & M^{*\perp}_x \\
B_X \arrow[u, "g_{\alpha\beta}(\tau)"'] & M^\perp_x \arrow[l, "P_{*x}|M^\perp_x"'] \arrow[u, "\gamma_{ab}(x)"']
\end{tikzcd} \qquad x = \beta(\tau)$$

Then we claim that $g_{\alpha\beta}(\tau)$ is the metric induced at X by the placement $\kappa_\tau = \psi_x \circ P^{-1}_{*x}$ where $x = \beta(\tau)$, regardless of the frame ψ_x. This is true since $g_{\alpha\beta}(\tau)W^\alpha V^\beta = \gamma_{ab}w^a(x)v^b(x)$ where $w^a = P^a_\alpha W^\alpha$, $v^b = P^b_\beta V^\beta$ are the isomorphism transformation images from B_X to $M^\perp_x$.

Proposition: *The alpha tensor* α^{ab} *is simply the lifted left Cauchy-Green tensor* $\hat{B}^{ab}$.

Proof: We write the deformation gradient $F_{r(X)}(\tau) = \psi_x \circ P^{-1}_{*x} \circ r^{-1}(X)$, and the left Cauchy-Green tensor $B = V^2 = FF^T$. Let $\underset{\sim}{v}$, $\underset{\sim}{w}$ be vectors in $\mathbb{R}^3$. The transpose or adjoint F^T of F is defined by the inner product condition $\langle \underset{\sim}{v}, F\underset{\sim}{w}\rangle = \langle F^T\underset{\sim}{v}, \underset{\sim}{w}\rangle$.

If we put the metric $g_{\alpha\beta}$ at X in B induced by $r(X)$, then the adjoint map can be defined for each of the maps above, and since both $r(X)$ and ψ_x are norm preserving they have adjoint and inverse equal, so that

$$F^T_{r(X)}(\tau) = [r^{-1}(X)]^+(P^{-1}_{*x})^+(\psi_x)^+ = r(X)(P^{-1}_{*x})^+\psi^{-1}_x .$$

Therefore $B = FF^T = \psi_x \circ P^{-1}_{*x} \circ (P^{-1}_{*x})^+ \circ \psi^{-1}_x$ which depends on $r(X)$ only through its induced metric at X so is unchanged if $r(X)$ is

replaced by $Q \circ r(X)$ where Q is orthogonal. This is frame indifferent and can be lifted to the tensor $\hat{B}$ at x. From the diagram shown here

$$\begin{tikzcd}
M_x^{\perp} \arrow[r, "(P_{*x}^{-1})^{+}"] \arrow[d, "\gamma_{ab}"] & B_X \arrow[d, "g_{\alpha\beta}"] \\
M_x^{\perp *} \arrow[d, "\alpha^{ab}"] & B_X^{*} \arrow[l, "P_x^{*}"'] \\
M_x^{\perp} \arrow[r, "P_{*x}"'] & B_X \arrow[u, "g_{\alpha\beta}"']
\end{tikzcd}$$

we see that $\quad P_{*x}^{-1} \circ (P_{*x}^{-1})^{+} = P_{*x}^{-1} \circ g_x^{-1} \circ g_x \circ (P_{*x}^{-1})^{+}$

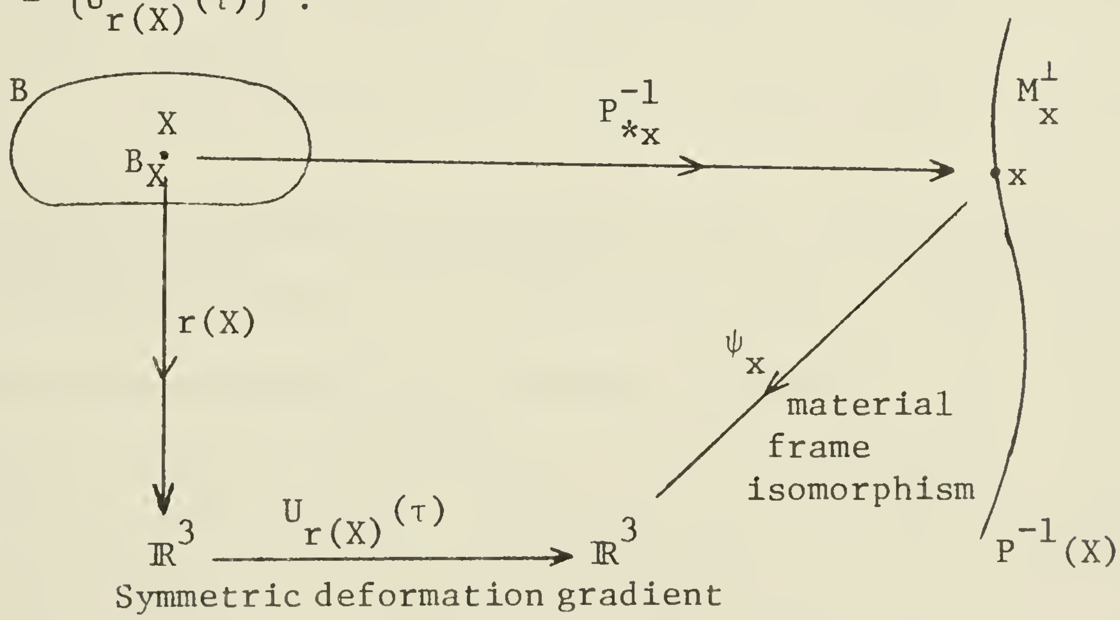
$$= \alpha : M_x^{\perp} \to M_x^{\perp}$$

where α is of the form $\alpha^{a}{}_{b}$ as an operator with one index lowered. (We may write $\alpha^{a}{}_{b} = \alpha_{b}^{a}$ since α is symmetric.) This shows us that the alpha tensor is in reality the left Cauchy-Green tensor for the reference placement $r(X)$. We have seen it is independent (through a rotation) of the orientation (not in the sense of Dieudonné [22], p. 150, but physically) of its reference placement, and is frame indifferent making it a tensor field on $P^{-1}(X)$. We can also obtain useful information about the right Cauchy-Green tensor which is simply a matrix (not a tensor on $P^{-1}(X)$) that is independent of the choice of frame (i.e. the ψ_x frame isomorphisms) but does on the other hand depend on the orientation (as above) of the reference placement, (i.e. is affected by $r(X) \to Qr(X)$, Q orthogonal). We have $C = F^{T}F = r(X)(P_{*x}^{-1})^{+}P_{*x}^{-1}r^{-1}(X)$. Through the reference placement $r(X)$ (which is infinitesimal) this corresponds to the transformation $g^{\alpha}{}_{\beta}(\tau) : B_X \to B_X$ where the reference induced metric is used to raise and lower indices and $g_{\alpha\beta}(\tau)$ is the

proper time dependent metric induced by the infinitesimal motion κ_τ. The diagram here, as before, commutes.

$$\begin{tikzcd}
M_x^{\perp} \arrow[r, "(P_{*x}^{-1})^{+}"] \arrow[d, "\gamma_{ab}"] & B_X \arrow[d, "g_{\alpha\beta}"] \\
M_x^{\perp *} & B_X^{*} \arrow[l, "P_x^{*}"'] \\
M_x^{\perp} \arrow[r, "P_{*x}"] \arrow[u, "\gamma_{ab}"] & B_X \arrow[u, "g_{\alpha\beta}(\tau)"']
\end{tikzcd}$$

The right Cauchy-Green tensor is also the square of the symmetric deformation gradient $F_{r(X)}(\tau) = U_{r(X)}(\tau)$ in a material frame, i.e. $C_{r(X)}(\tau) = \left(U_{r(X)}(\tau)\right)^2$.



(I.10) <u>The Colon Contorsion - Convective Derivative in Lie Derivative Form.</u>

We can define a particular contorsion tensor K_{abc} as follows. Put $K_{abc} = (u_a\dot{u}_b - u_b\dot{u}_a)u_c$ where $\dot{u}_a = u_{a;b}u^b$ and ; is Christoffel symbol covariant differentiation. Clearly this contorsion (and the corresponding connection) is metric and orthogonally invariant. Furthermore if we let ":" denote the associated covariant differentiation, then $u^a{}_{:b} = u^a{}_{,b} + u^c\Gamma_c{}^a{}_b = u^a{}_{,b} + u^c\left(\left\{{a \atop c\ b}\right\} - K_c{}^a{}_b\right) = u^a{}_{;b} - u^c K_c{}^a{}_b$. Hence lowering indices,

and using $g_{ab;c} = g_{ab:c} = 0$ we have,

$$u_{a:b} = u_{a;b} - u^c K_{cab} = u_{a;b} + \dot{u}_a u_b \quad \text{since} \quad u^c \dot{u}_c = 0$$

$$= \theta_{ab} + \omega_{ab} \quad \text{(following Carter and Quintana [13])},$$

where $\theta_{ab} = u_{(a:b)}$ and $\omega_{ab} = u_{[a:b]}$. It is easy to see that $u_{a:b}u^b = 0 = u_{b:a}u^b$ so that θ_{ab} and ω_{ab} are orthogonal. Also the flow lines are autoparallels with respect to this connection since $u_{a:b}u^b = 0$. We shall see later that this connection produces a spatially non-rotating frame like the Christoffel symbols (or Lévi-Civita connection).

The Lie derivative of a tensor field $T^{a\cdots}{}_{b\cdots}$ can be written in component form as

$$\begin{aligned} \mathcal{L}_{\underset{\sim}{u}} T^{a\cdots}{}_{b\cdots} &= T^{a\cdots}{}_{b\cdots,c}u^c - T^{c\cdots}{}_{b\cdots}u^a{}_{,c} - \cdots + T^{a\cdots}{}_{c\cdots}u^c{}_{,b} + \cdots \\ &= T^{a\cdots}{}_{b\cdots;c}u^c - T^{c\cdots}{}_{b\cdots}u^a{}_{;c} - \cdots + T^{a\cdots}{}_{c\cdots}u^c{}_{;b} + \cdots \end{aligned} \qquad \text{(I.10.1)}$$

where , denotes partial differentiation and ; Christoffel symbol covariant differentiation. If we replace ; by : above then it is easy to see that we obtain the convective derivative, i.e.

$$\mathcal{D}T^{a\cdots}{}_{b\cdots} = T^{a\cdots}{}_{b\cdots:c}u^c - T^{c\cdots}{}_{b\cdots}u^a{}_{:c} - \cdots + T^{a\cdots}{}_{c\cdots}u^c{}_{:b} + \cdots \qquad \text{(I.10.2)}$$

Furthermore if $T^{a\cdots}{}_{b\cdots}$ is an orthogonal tensor field it is clear that each term in the above expression is orthogonal since $u_{a:b}$ is orthogonal. We say that by means of the connection associated with the covariant derivative :, we have reduced the convective derivative to Lie derivative form. The general requirement on a contorsion for this to be true is $u_c\dot{u}_b = u^a(K_{bca} - K_{acb})$. Also one can show that the only metric preserving covariant differentiation with this property (of producing the convective derivative in Lie derivative form) which is

orthogonally invariant with respect to the Christoffel symbols is the colon covariant derivative. For if $K_{bca} = -B_{bc}u_a$ is the contorsion for such a connection, and if we put $\Delta B_{bc} = B_{bc} - (\dot{u}_b u_c - u_b \dot{u}_c)$ then $0 = \Delta B_{bc} + u^a \Delta B_{ac} u_b$ which implies $\Delta B_{bc} = 0$ if ΔB_{bc} is antisymmetric.

(I.11) The Volume Element in Four Dimensions.

The Lorentz metric g_{ab} on M defines for us a volume element (a completely skew smooth covariant tensor field on M of rank 4). We shall see later that M, which is that portion of space time through which the material medium moves, is an orientable manifold. This follows from the fact that B is orientable and the Lorentz group structure on M is orthochronous (see Chapter II) and the notion of future pointing is well defined, with $\underset{\sim}{u}$ a future pointing vector field on all of M. This permits us to define the metric volume element on all of M independent of ambiguity in the sign. In local coordinates we write $\underline{\varepsilon}^* = 4!\sqrt{g}\, dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 = \varepsilon_{ijk\ell} dx^i \otimes dx^j \otimes dx^k \otimes dx^\ell = \varepsilon_{ijk\ell} dx^i \wedge dx^j \wedge dx^k \wedge dx^\ell$. Here $\varepsilon_{ijk\ell} = \sqrt{g}\, \epsilon_{ijk\ell}$ where $\epsilon_{1234} = 1$ and ϵ is completely skew. Of course $g = -\det(g_{ij})$ in the specified coordinate system. Dual to $\underline{\varepsilon}^*$ we have $\underline{\varepsilon} = -\dfrac{4!}{\sqrt{g}} \dfrac{\partial}{\partial x^1} \wedge \dfrac{\partial}{\partial x^2} \wedge \dfrac{\partial}{\partial x^3} \wedge \dfrac{\partial}{\partial x^4} = \varepsilon^{ijk\ell} \dfrac{\partial}{\partial x^i} \otimes \dfrac{\partial}{\partial x^j} \otimes \dfrac{\partial}{\partial x^k} \otimes \dfrac{\partial}{\partial x^\ell} = \varepsilon^{ijk\ell} \dfrac{\partial}{\partial x^i} \wedge \dfrac{\partial}{\partial x^j} \wedge \dfrac{\partial}{\partial x^k} \wedge \dfrac{\partial}{\partial x^\ell}$. Let $\epsilon^{ijk\ell} = \epsilon_{ijk\ell}$. Then $\varepsilon^{ijk\ell} = -\dfrac{\epsilon^{ijk\ell}}{\sqrt{g}}$, $\varepsilon^{abcd}\varepsilon_{abcd} = -4!$, and $\varepsilon_{ijk\ell} = g_{ia}g_{jb}g_{kc}g_{\ell d}\varepsilon^{abcd}$. Likewise we have a volume element on the body manifold, namely $\overset{*}{\eta}_{\alpha\beta\gamma}$ where Greek indices run from 1 to 3 and refer to the body, and Latin indices from 1 to 4 refer to space time. Then $\underline{\eta}^* = 3!\bar{\eta}(X)dX^1 \wedge dX^2 \wedge dX^3 = \overset{*}{\eta}_{\alpha\beta\gamma}(X)dX^\alpha \otimes dX^\beta \otimes dX^\gamma$. $\overset{*}{\eta}_{\alpha\beta\gamma}$ is completely skew symmetric and $\overset{*}{\eta}_{123} = \bar{\eta}$. The mass of a subset A of B in a local placement ϕ is

$m(A) = \int_{\phi(A)} \rho\, dV = \int_{\phi(A)} \bar{\eta}(X)dX^1dX^2dX^3$. ϕ is the coordinate map from $V \subset B$ into $\mathbb{R}^3$, $\phi(X) = (X^1,X^2,X^3)$ and $A \subset V$ is a measurable set. Hence the tensor density $\bar{\eta}$ determines (and is) the mass density ρ in the placement ϕ, i.e. $\rho = \bar{\eta}(X)$.

We have the standard duality for the volume element, namely

$$\underline{\eta} = \frac{3!}{\bar{\eta}(X)} \frac{\partial}{\partial X^1} \wedge \frac{\partial}{\partial X^2} \wedge \frac{\partial}{\partial X^3} = \eta^{\alpha\beta\gamma}(X) \frac{\partial}{\partial X^\alpha} \otimes \frac{\partial}{\partial X^\beta} \otimes \frac{\partial}{\partial X^\gamma}$$

(Wang [108], p. 44). The $\eta^{\alpha\beta\gamma}$ are completely skew and moreover $\eta^{123} = \frac{1}{\bar{\eta}} = \frac{1}{\overset{*}{\eta}_{123}}$ so $\overset{*}{\eta}_{\alpha\beta\gamma}\eta^{\alpha\beta\gamma} = 3!$. Then $\eta^{\alpha\beta\gamma}$, $\overset{*}{\eta}_{\alpha\beta\gamma}$ can be lifted uniquely to materially constant orthogonal space time completely skew tensor fields on M which we denote by $\hat{\eta}^{abc}$ and $\hat{\overset{*}{\eta}}_{abc}$ respectively. Then $\mathcal{D}\hat{\eta}^{abc} = 0$ and $\mathcal{D}\hat{\overset{*}{\eta}}_{abc} = 0$. We use the maps $\overset{*}{P}_x$ and $(P_{*x}|M_x^{\perp})^{-1}$ in order to construct these tensors for each $x \in M$.

On M let us use the metric γ_{ab} to lower indices (for orthogonal tensors) at each x, and on B let us use $g_{\alpha\beta}(\tau)$ corresponding to $x = \beta(\tau) \in P^{-1}(X)$. Recall that this is the metric induced by the placement κ_τ at the point X at proper time τ. These two metrics transform isomorphically to one another through P_* as we have seen.

We can find a real valued function $\lambda(X,\tau)$ such that

$$\lambda(X,\tau)\eta^{\rho\sigma\mu}(X)g_{\rho\alpha}(X,\tau)g_{\sigma\beta}(X,\tau)g_{\mu\gamma}(X,\tau) = \overset{*}{\eta}_{\alpha\beta\gamma}(X).$$

Of course we have the natural determinism $x = x(X,\tau)$ of the point $x \in M$. The proper time parametrization is selected arbitrarily for each X and there is no need of a special relationship even such as continuity of $x(X,\tau)$ for fixed τ as X varies. We also write $\lambda(x) = \lambda(X,\tau)$ where $x = x(X,\tau)$. Then through the P_* lifting we have $\lambda(x)\hat{\eta}^{psu}(x)\gamma_{pa}(x)\gamma_{sb}(x)\gamma_{uc}(x) = \hat{\overset{*}{\eta}}_{abc}(x)$ for $x \in M$. The mass density

ρ is given by $\rho(X,\tau) = \rho(x) = \overset{*}{\eta}_{123}\, J$ where $J = \det(\phi_{*X} \circ \kappa_\tau^{-1})$ where ϕ is the body coordinate map. Clearly $\rho(x)$ is independent of the choice of coordinates on B at X. For the sake of the one point X (now fixed) and proper time τ (also fixed) we choose $V \subset B$ with $x \in V$ and $\phi\colon V \to \mathbb{R}^3$ a diffeomorphism onto an open subset, with $\phi(X) = (X^1,X^2,X^3)$ such that at this particular (X,τ), $\phi_{*X} = \kappa_\tau$. Then $\rho = \overset{*}{\eta}_{123}$ and at (X,τ), $\lambda\eta^{\rho\sigma\mu} g_{\rho\alpha} g_{\sigma\beta} g_{\mu\gamma} = \overset{*}{\eta}_{\alpha\beta\gamma}$. With these coordinates at that point $g_{\rho\sigma}$ is the identity 3×3 matrix, so $\lambda\eta^{\alpha\beta\gamma} = \overset{*}{\eta}_{\alpha\beta\gamma}$ or $\lambda\eta^{123} = \overset{*}{\eta}_{123}$. Hence $\lambda = \dfrac{\overset{*}{\eta}_{123}}{\eta^{123}} = \overset{*}{\eta}{}^{2}_{123} = \eta^{-2} = \rho^2$, i.e. $\lambda = \rho^2$. Therefore we see that $\hat{\overset{*}{\eta}}_{abc} = \rho^2 \hat{\eta}_{abc}$ where $\hat{\eta}_{abc}$ is obtained from $\hat{\eta}^{abc}$ by lowering of indices using the γ_{ab} tensor (equivalently using g_{ab} since $\hat{\eta}^{abc}$ is orthogonal and $g_{ab} = \gamma_{ab} - u_a u_b$).

Also we must have $\overset{*}{\underset{\sim}{\varepsilon}} = \overset{*}{\sigma}\, \hat{\overset{*}{\underset{\sim}{\eta}}} \wedge \overset{*}{\underset{\sim}{u}}$ and $\underset{\sim}{\varepsilon} = \sigma \hat{\underset{\sim}{\eta}} \wedge \underset{\sim}{u}$ where $\overset{*}{\sigma}$, σ are scalar fields, $\underset{\sim}{u}$ has components u^a and $\overset{*}{\underset{\sim}{u}}$ has components u_a. Using g_{ab} to lower indices on this latter equation we have $\overset{*}{\underset{\sim}{\varepsilon}} = \dfrac{\sigma}{\rho^2} \hat{\overset{*}{\underset{\sim}{\eta}}} \wedge \overset{*}{\underset{\sim}{u}}$ so that $\overset{*}{\sigma} = \sigma/\rho^2$. In components

$$\varepsilon^{ijk\ell} = \frac{\sigma}{4}\left(\hat{\eta}^{ijk}u^{\ell} + \hat{\eta}^{jk\ell}u^{i} + \hat{\eta}^{k\ell i}u^{j} + \hat{\eta}^{\ell ij}u^{k}\right).$$

Now at x we choose properly comoving coordinates (Carter and Quintana [13]) with $u^1 = u^2 = u^3 = 0$ and $u^4 = 1$ so $\varepsilon^{1234} = \frac{\sigma}{4}\hat{\eta}^{123}$. If $x = x(X,\tau)$ is fixed the coordinates can also be taken so that $\phi_{*X} = \kappa_\tau$ where $\phi(X) = (X^1,X^2,X^3)$ comove with (x^1,x^2,x^3,x^4) i.e. $X^1 = x^1$, $X^2 = x^2$, $X^3 = x^3$ and $x^4 = \tau$. The condition $\phi_{*X} = \kappa_\tau$ holds only at x. Hence at this point, $\hat{\eta}^{123} = \dfrac{1}{\hat{\overset{*}{\eta}}_{123}} = \dfrac{1}{\rho}$, but $\varepsilon^{1234} = -\dfrac{1}{\sqrt{g}}$ where $g = -\det(g_{ij})$ in this coordinate system. The metric g_{ij} is clearly $\mathrm{diag}(1,1,1,-1)$ at x so $g = 1$ and $\varepsilon^{1234} = -1$. Hence $-1 = \dfrac{\sigma}{4\rho}$ so $\sigma = -4\rho$ and $\overset{*}{\sigma} = -4/\rho$. Therefore $\underset{\sim}{\varepsilon} = 4\rho \underset{\sim}{u} \wedge \hat{\underset{\sim}{\eta}}$ and $\overset{*}{\underset{\sim}{\varepsilon}} = \dfrac{4}{\rho} \overset{*}{\underset{\sim}{u}} \wedge \hat{\overset{*}{\underset{\sim}{\eta}}}$. Since $\underset{\sim}{u} \wedge \hat{\underset{\sim}{\eta}}$ is convective invariant we have

$$\mathcal{D}\underset{\sim}{\varepsilon} = 4\underset{\sim}{u} \wedge \hat{\underset{\sim}{\eta}}\left(\frac{d\rho}{d\tau}\right) = \underset{\sim}{\varepsilon}\,\frac{d}{d\tau}\,(\ln\rho),$$

$$\mathcal{D}\underset{\sim}{\varepsilon}^{*} = -\frac{4}{\rho^2}\,\underset{\sim}{u}^{*} \wedge \hat{\underset{\sim}{\eta}}^{*}\left(\frac{d\rho}{d\tau}\right) = -\underset{\sim}{\varepsilon}^{*}\,\frac{d}{d\tau}\,(\ln\rho).$$

Notice that $\varepsilon^{ijk\ell}u_\ell = \frac{\sigma}{4}\hat{\eta}^{ijk}{}_{u}u^{\ell}u_{\ell} = \rho\hat{\eta}^{ijk}$. Lowering indices we get $\varepsilon_{ijk\ell}u^{\ell} = \frac{1}{\rho}\,\hat{\eta}^{*}_{ijk}$. We write (as a matter of notation) $\varepsilon^{ijk} = \varepsilon^{ijk\ell}u_\ell$ and $\varepsilon_{ijk} = \varepsilon_{ijk\ell}u^{\ell}$. Hence for these orthogonal 3-volume elements $\varepsilon_{ijk} = \frac{1}{\rho}\,\hat{\eta}^{*}_{ijk}$ and $\varepsilon^{ijk} = \rho\hat{\eta}^{ijk}$. Also we see that $\hat{\eta}^{*}_{ijk}\varepsilon^{ijk} = 3!\rho = 6\rho$.

<u>Proposition</u>: *If* $|$ *is a metric covariant derivative,* $\varepsilon_{abcd|e} = 0$.

<u>Proof</u>: Recall $\varepsilon_{abcd} = \sqrt{g}\,\epsilon_{abcd}$. Now,

$$\varepsilon_{abcd;e} = \varepsilon_{abcd,e} - \varepsilon_{fbcd}\left\{{f \atop a\;e}\right\} - \varepsilon_{afcd}\left\{{f \atop b\;e}\right\} - \varepsilon_{abfd}\left\{{f \atop c\;e}\right\} - \varepsilon_{abcf}\left\{{f \atop d\;e}\right\}$$

so $\varepsilon_{1234;e} = (\sqrt{g})_{,e} - \left\{{a \atop a\;e}\right\}\sqrt{g} = (\sqrt{g})_{,e} - \sqrt{g}(\ln\sqrt{g})_{,e} = 0$ since $\left\{{a \atop a\;e}\right\} = (\ln\sqrt{g})_{,e}$ is easily verified (with sum on a) using $\left\{{a \atop b\;c}\right\} = \frac{1}{2}\,g^{ad}(g_{db,c} + g_{dc,b} - g_{bc,d})$. Then with a contorsion,

$$\varepsilon_{abcd|e} = \varepsilon_{abcd;e} + \varepsilon_{fbcd}K_a{}^f{}_e + \varepsilon_{afcd}K_b{}^f{}_e + \varepsilon_{abfd}K_c{}^f{}_e + \varepsilon_{abcf}K_d{}^f{}_e.$$

Thus $\varepsilon_{1234|e} = \varepsilon_{1234}K_a{}^a{}_e = 0$ since $K_a{}^a{}_e = g^{ab}K_{abe} = 0$ because $K_{abe} = K_{[ab]e}$ for a metric (or Lorentz) contorsion. Therefore $\varepsilon_{abcd|e} = 0$ as required. By raising indices we can see that $\varepsilon^{abcd}{}_{|e} = 0$ also, and the proof is complete.

Recall that $u_{a:b} = u_{a;b} + \dot{u}_a u_b$ and $\theta_{ab} = u_{(a:b)}$ and $\omega_{ab} = u_{[a:b]}$. If we put $\theta_{ab}\gamma^{ab} = \theta_{ab}g^{ab} = \theta$ we can see that $\theta = u^a{}_{:a} = u^a{}_{;a}$. Also $\mathcal{D}\gamma_{ab} = \mathcal{D}(g_{ab} + u_a u_b) = \mathcal{D}g_{ab} = g_{ab:e}u^e + g_{eb}u^e{}_{:a} + g_{ae}u^e{}_{:b} = u_{b:a} + u_{a:b} = 2\theta_{ab}$. Thus $\mathcal{D}\gamma_{ab} = 2\theta_{ab}$ and similarly $\mathcal{D}\gamma^{ab} = -2\theta^{ab}$, $\mathcal{D}\gamma^a_b = 0$.

Now $\mathcal{D}\varepsilon_{abcd} = \varepsilon_{abcd:e}u^e + \varepsilon_{ebcd}u^e{}_{:a} + \varepsilon_{aecd}u^e{}_{:b} + \varepsilon_{abed}u^e{}_{:c} + \varepsilon_{abce}u^e{}_{:d} = f\varepsilon_{abcd}$ where f is a scalar function of position, since

the tensor is completely skew and of maximal rank. Hence $f\varepsilon_{1234} = \varepsilon_{1234}u^a{}_{:a}$ so $f = \theta$ and $\mathcal{D}\varepsilon_{abcd} = \theta\varepsilon_{abcd}$. Comparing with our previous result $\mathcal{D}\underset{\sim}{\varepsilon}^* = -\underset{\sim}{\varepsilon}^* \frac{d}{d\tau}(\ln\rho)$ we find that $\theta = -\frac{d}{d\tau}(\ln\rho)$. Thus θ is a measure of the proper time expansion rate of the material along world lines. By substituting back into the result $\mathcal{D}\underset{\sim}{\varepsilon} = \underset{\sim}{\varepsilon}\frac{d}{d\tau}(\ln\rho)$ we obtain $\mathcal{D}\varepsilon^{abcd} = -\theta\varepsilon^{abcd}$. Likewise $\mathcal{D}\varepsilon^{abc} = -\theta\varepsilon^{abc}$, $\mathcal{D}\varepsilon_{abc} = \theta\varepsilon_{abc}$, since $\mathcal{D}\underset{\sim}{u} = 0$. The equation $\theta = -\frac{d}{d\tau}(\ln\rho)$ can be written as $\theta\rho + \frac{d\rho}{d\tau} = \theta\rho + \dot{\rho} = 0$, so $(\rho u^a)_{:a} = (\rho u^a)_{;a} = 0$. This is the continuity equation which expresses local conservation of mass in that there is no local flux of mass through the surface of a fixed sphere not accounted for by a density change.

(I.12) Relative Deformation and Rate of Change.

Recall that the relative deformation $F_\tau(\tau')$ satisfies $F_\tau(\tau') \circ F_{r(X)}(\tau) = F_{r(X)}(\tau')$ and transforms like $\overline{F}_\tau(\tau') = Q(\tau')F_\tau(\tau')Q^T(\tau)$ under a change of frame. Following Truesdell [105], p. 19 we introduce the analogs to what he calls stretching and spin, using the same symbols for convenience. We let $G = \dot{F}_\tau(\tau) = \frac{d}{d\tau'} F_\tau(\tau')\Big|_{\tau'=\tau}$ and in the bar frame, $\overline{G} = \dot{\overline{F}}_\tau(\tau) = \frac{d}{d\tau'}\overline{F}_\tau(\tau')\Big|_{\tau'=\tau}$. Also, we can decompose F_τ (as a function of τ') in the usual way, namely $F_\tau = R_\tau U_\tau = V_\tau R_\tau$, $C_\tau = U_\tau^2$, $B_\tau = V_\tau^2$ where C_τ and B_τ are the relative right and left Cauchy-Green tensors respectively, R_τ is the relative rotation, and U_τ and V_τ the relative right and left stretches. We then define $D = \dot{U}_\tau(\tau) = \dot{V}_\tau(\tau)$ and $W = \dot{R}_\tau(\tau)$, and it follows easily that $D^T = D$, $W^T + W = 0$, $G = \dot{F}_{r(X)}(\tau)F_{r(X)}^{-1}(\tau) = D + W$. Just differentiate $F_\tau(\tau') = R_\tau(\tau')U_\tau(\tau')$ with respect to τ' and put $\tau' = \tau$ to get these results. Also [105, p. 21], (P.3.1),

$$W = \dot{R}\,R^T + \frac{1}{2}\,R(\dot{U}\,U^{-1} - U^{-1}\,\dot{U})R^T \quad \text{and}$$

$$D = \frac{1}{2}\,R(\dot{U}\,U^{-1} + U^{-1}\,\dot{U})R^T$$

where $R = R_{r(X)}(\tau)$ and $\dot{R} = \frac{d}{d\tau} R_{r(X)}(\tau)$, $U = U_{r(X)}(\tau)$, $RU = F_{r(X)}(\tau)$ etc. Also we define the Rivlin-Ericksen tensors A_n by

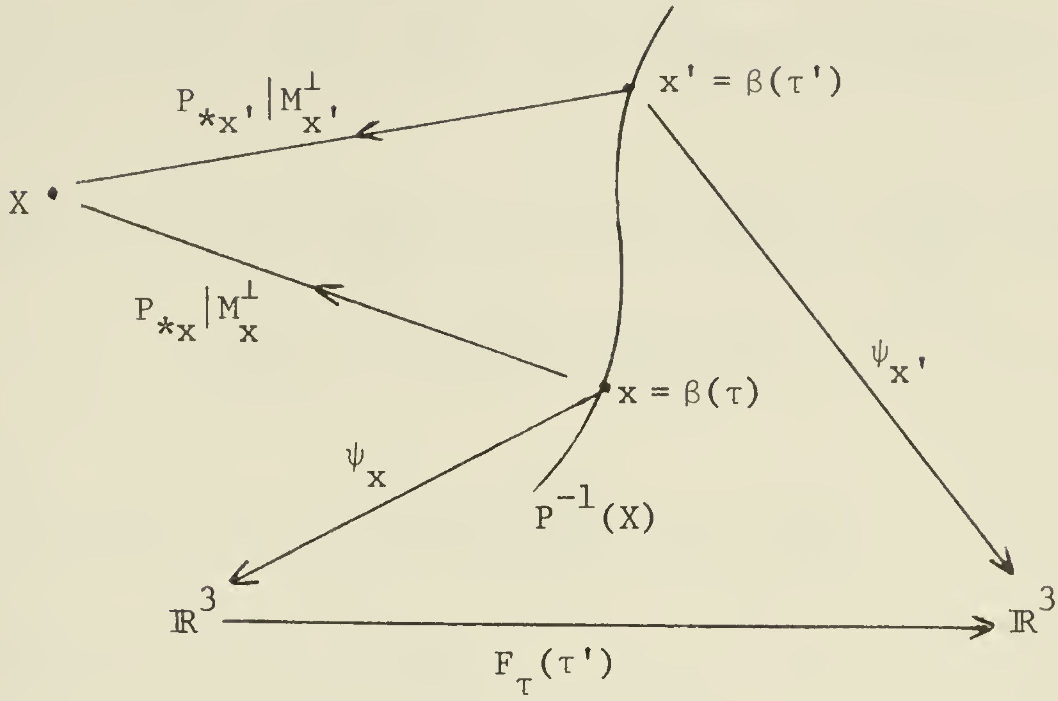
$$A_n = C_\tau^{[n]}(\tau) = \frac{d^n}{d\tau'^n} C_\tau(\tau')\Big|_{\tau'=\tau} .$$

Let us now consider a change of frame and see how these defined quantities transform. Recall earlier we defined $\Lambda = Q^{-1}\dot{Q}$. Now we introduce $A = \dot{Q}Q^T = \dot{Q}Q^{-1} = Q\Lambda Q^T$. Both Λ and A are antisymmetric 3×3 matrix functions of τ along $P^{-1}(X)$. If we take $F = F_{r(X)}(\tau)$ we see that under a change of frame $\bar{F} = QF$, $\bar{R} = QR$, $\bar{U} = U$, $\bar{V} = QVQ^T$ so that differentiation with respect to τ gives $\dot{\bar{F}} = \dot{Q}F + Q\dot{F}$. But $\dot{F} = GF$ and $\dot{\bar{F}} = \bar{G}\bar{F}$ so $\bar{G}\bar{F} = QGF + \dot{Q}F = QGQ^T\bar{F} + \dot{Q}Q^T\bar{F}$. Multiplying to the right by $\bar{F}^{-1}$ gives $\bar{G} = QGQ^T + A$, or separating symmetric and antisymmetric parts gives $\bar{D} = QDQ^T$ and $\bar{W} = QWQ^T + A$. In particular D is frame indifferent and can be lifted to a symmetric tensor $\hat{\underset{\sim}{D}}$ on M which is independent of any frame or reference placement. It is also not hard to show that the Rivlin-Ericksen tensors are frame indifferent, i.e. $\bar{A}_n = QA_nQ^T$ and therefore have frame and reference independent lifts $\hat{\underset{\sim}{A}}_n$ to M. Notice that $A_1 = \frac{d}{d\tau'} C_\tau(\tau')\Big|_{\tau'=\tau}$ and $C_\tau(\tau') = U_\tau^2(\tau')$ so $A_1 = 2D$ and $\hat{\underset{\sim}{A}}_1 = 2\hat{\underset{\sim}{D}}$.

(I.13) <u>Evaluation of the Rivlin-Ericksen Tensors</u>.

Recall that $F_\tau(\tau') = \psi_{x'} \circ P^{-1}_{*x'} \circ P_{**} \circ \psi_x^{-1}$ and $C_\tau(\tau') = F_\tau^T(\tau')F_\tau(\tau')$. In order to calculate $F_\tau^T(\tau')$ we have to get its adjoint which necessitates putting a metric at X. We choose $g_{\alpha\beta}(\tau')$ so that

$\psi_{x'}$, ψ_x, and $P_{*x'}$ are norm and inner product preserving meaning their adjoints equal their inverses.



Now $C_\tau(\tau') = F^{\mathsf{T}}_\tau(\tau') \circ F_\tau(\tau') = (\psi_x \circ P^{+}_{*x} \circ P_{*x'} \circ \psi^{-1}_{x'}) \circ (\psi_{x'} \circ P^{-1}_{*x'} \circ P_{*x} \circ \psi^{-1}_{x}) = \psi_x \circ P^{+}_{*x} \circ P_{*x} \circ \psi^{-1}_{x}$. The τ' dependence is completely contained here in the adjoint (+) operation.

$$
\begin{tikzcd}
M^{\perp}_{x} \arrow[d, "\gamma_{ab}"] & B_X \arrow[l, "P^{+}_{*x}"'] \arrow[d, "g_{\alpha\beta}(\tau')"] \\
M^{*\perp}_{x} & B^{*}_{X} \arrow[l, "P^{*}_{x}"'] \\
M^{\perp}_{x} \arrow[u, "\beta_{ab}"] \arrow[r, "P_{*x}"] & B_X \arrow[u, "g_{\alpha\beta}(\tau')"']
\end{tikzcd}
$$

Reference placement determining β_{ab}
is $\kappa_{\tau'}$ inducing metric $g_{\alpha\beta}(\tau')$.

From the commutative diagram here we see that $P^{+}_{*x} \circ P_{*x} : M^{\perp}_{x} \to M^{\perp}_{x}$ is determined by $\beta^{a}{}_{b}$, the beta tensor for the reference placement $\kappa_{\tau'}$.

$\beta_{ab} = \gamma_{ab}$ at x' and β_{ab} is obtained on $P^{-1}(X)$ by convective transport of $\gamma_{ab}(x')$ along all of $P^{-1}(X)$.

Noticing that since $\frac{d}{d\tau'}$ has no effect on functions of x only, we can write,

$$\frac{d}{d\tau'} C_\tau(\tau') = \psi_x \circ \left(\frac{d}{d\tau'} [P^+_{*x} \circ P_{*x}]\right) \circ \psi_x^{-1} = \psi_x \circ \left[\frac{d}{d\tau'} (\beta^a{}_b)\right] \circ \psi_x^{-1}$$
$$= \psi_x \circ \left(\gamma^{ac}(x) \frac{d}{d\tau'} (\beta_{cb})\right) \circ \psi_x^{-1} .$$

Also,

$$\frac{d^n}{d\tau'^n} C_\tau(\tau') = \psi_x \circ \left(\gamma^{ac}(x) \frac{d^n}{d\tau'^n} (\beta_{cb})\right) \circ \psi_x^{-1}.$$

Evaluating at $\tau' = \tau$ we have

$$A_{(n)} = \psi_x \circ \left(\gamma^{ac}(x) \mathcal{D}^n \gamma_{cb}\right) \circ \psi_x^{-1}$$

where $\mathcal{D}^n$ is the n-fold convective derivative. Hence $\hat{A}_{(n)}{}^{ad} = \gamma^{ac}(\mathcal{D}^n \gamma_{cb})\gamma^{bd}$ and $\hat{A}_{(1)}{}^{ad} = 2\theta^{ad}$ so $\hat{D}^{ad} = \theta^{ad}$. This gives us the physical interpretation of actually what the tensor θ^{ad} represents in relativistic continuum mechanics. It is the instantaneous relative symmetric deformation rate along flow lines. A motion with $\theta = 0$ we call *isochoric* and with $\theta_{ab} = 0$ we call *rigid*. (Compare this with Söderholm [95].)

(I.14) The Materially Nonrotating Frame.

Recall that $\overline{W} = QWQ^T + A = Q(W + \Lambda)Q^T$. If we choose our frame transformation Q such that $W = -\Lambda$ then we obtain $\overline{W} = 0$ so the new transformed frame is *materially nonrotating*. Clearly for any anti-symmetric matrix function W of τ along $P^{-1}(X)$ we can choose an orthogonal matrix function $Q(\tau)$ so as to make $\Lambda(\tau) = Q^T\dot{Q}$ equal to $-W(\tau)$. The function $Q(\tau)$ is unique up to a fixed τ independent

orthogonal left multiplication i.e. $Q_0 Q(\tau)$ does just as well, Q_0 orthogonal, $\det Q_0 = +1$. This tells us that every materially nonrotating frame (i.e. orthonormal triad on $M_x^{\perp}$ for each $x \in P^{-1}(X)$, smoothly determined, nonrotating materially as above) is related to any other materially nonrotating frame on $P^{-1}(X)$ by a fixed τ independent orthogonal matrix Q_0 transforming frames at each point $x \in P^{-1}(X)$. Also each Q_0 transforms one materially nonrotating frame to another.

(I.15) Materially Nonrotating Connections and Covariant Derivatives.

We seek a covariant derivative $|$ and a corresponding contorsion k_{abc} which is metric, that will represent the materially non-rotating frame under a Fermi Transport. First let us suppose this contorsion reduces the convective derivative to Lie derivative form. If v^a is a Fermi transport invariant vector field on $P^{-1}(X)$ with respect to the covariant differentiation $|$ then $\frac{\delta v^a}{d\tau} = u^a\left(\frac{\delta u_b}{d\tau}\right)v^b$. If v^a is defined on a neighborhood of $P^{-1}(X)$, $v^a{}_{|b}u^b = u^a u_{b|c}u^c v^b$. Hence, $\mathcal{D}v^a = v^a{}_{|b}u^b - u^a{}_{|b}v^b$. But the condition $u_c\dot{u}_b = u^a(K_{bca} - k_{acb})$ for Lie derivative form automatically implies $u_{b|c}u^c = 0$ for metric contorsions. Hence, $\mathcal{D}v^a = -u^a{}_{|b}v^b$ for a Fermi transported field v^a. Now let $\underset{\sim}{r}$, $\underset{\sim}{s}$, $\underset{\sim}{t}$ be vector fields on $P^{-1}(X)$ determining an orthonormal frame in $M_x^{\perp}$ for each $x \in P^{-1}(X)$ that is Fermi transported along $P^{-1}(X)$ as above, i.e. $\mathcal{D}r^a = -u^a{}_{|b}r^b$, $\mathcal{D}s^a = -u^a{}_{|b}s^b$, $\mathcal{D}t^a = -u^a{}_{|b}t^b$. Suppose ψ_x is the frame isomorphism determining these fields, i.e. $\psi_x\big(\underset{\sim}{r}(x)\big) = \underset{\sim}{e}_{(1)}$, $\psi_x\big(\underset{\sim}{s}(x)\big) = \underset{\sim}{e}_{(2)}$, $\psi_x\big(\underset{\sim}{t}(x)\big) = \underset{\sim}{e}_{(3)} \in \mathbb{R}^3$, $\forall\, x \in P^{-1}(X)$. Then by looking at $F_\tau(\tau') = \psi_{x'} \circ P^{-1}_{*x'} \circ P_{*x} \circ \psi_x^{-1}$ we see that for an infinitesimal transformation at x,

$$F_\tau(\tau + d\tau) = I + d\tau \begin{bmatrix} u_{c|b} r^c r^b & u_{c|b} r^c s^b & u_{c|b} r^c t^b \\ u_{c|b} s^c r^b & u_{c|b} s^c s^b & u_{c|b} s^c t^b \\ u_{c|b} t^c r^b & u_{c|b} t^c s^b & u_{c|b} t^c t^b \end{bmatrix} ,$$

where I is the 3×3 identity matrix. Hence the condition for $W = 0$ is $u_{c|b} = u_{b|c}$. The two conditions on the contorsion can thus be written $u_c \dot{u}_b = u^a(K_{bca} - K_{acb})$ [Lie derivative form] and $u_{a;b} - u_{b;a} = u^c(K_{bca} - K_{acb})$ [Symmetry $u_{c|b} = u_{b|c}$]. The simplest (but not the only) solution is found by taking $K_{bca} - K_{acb} = (u_{b;a} - u_{a;b})u_c$, and using $K_{bca} + K_{cba} = 0$ we find that

$$2K_{bca} = (u_{b,c} - u_{c,b})u_a + (u_{a,c} - u_{c,a})u_b + (u_{b,a} - u_{a,b})u_c ,$$

where the partial derivative "," may be replaced by covariant derivative ";". We can then easily check that $u_{a|b} = \theta_{ab}$ for this contorsion, and we denote this covariant derivative by "!" so $u_{a!b} = \theta_{ab}$.

If we look for an orthogonally invariant materially nonrotating contorsion, we have to drop the condition that the convective derivative be in Lie derivative form, otherwise no solutions are found. Let $\Gamma^a_{b\,c} = \left\{ {a \atop b\,c} \right\} - K^{\,a}_{b\,c}$ where $K_{bac} = -B_{ba}u_c$, $B_{ba} + B_{ab} = 0$. If Γ is a $\underset{\sim}{u}$-autoparallel connection $(u_{a|b}u^b = 0)$ then $\dot{u}_a = -B_{ac}u^c$ and $u_{a|b} = u_{a;b} + \dot{u}_a u_b$. Fermi invariance of a field w^a implies $\frac{\delta w^a}{d\tau} = 0$, and so $g_{ab}\mathcal{D}w^b = -w^b(u_{(a;b)} + u_{(a}\dot{u}_{b)}) = -w^b\theta_{ab}$ for the materially nonrotating condition means $K_{abc} = (u_{[a;b]} + u_{[a}\dot{u}_{b]})u_c$.

It is easy to see that for a $\underset{\sim}{u}$-autoparallel connection, the Fermi transports along the flows equal the connection parallel transports. Also we can see that an orthogonally invariant contorsion $K_{abc} = -B_{ab}u_c$ gives rise to the same Fermi transport frames as the one obtained by orthogonalizing B_{ab}, (i.e. replacing it by $\gamma^c_a\gamma^d_b B_{cd}$ where $\gamma^c_a = \delta^c_a + u^c u_a$).

The frames determined by Christoffel symbol Fermi transport along $P^{-1}(X)$ (unique up to constant orthogonal matrix transformation) are called *spatially nonrotating*. This is justified on physical grounds by Enosh and Kovetz [27] and Lianis [50] pp. 62-64. The colon contorsion with covariant derivative ":" is also spatially nonrotating with the same Fermi frames (= parallel transport frames in this case) as the Christoffel symbol or semicolon derivative with zero contorsion. The condition on a contorsion for it to be materially nonrotating is $\gamma^a_d \gamma^b_e K_{abc} u^c = -\omega_{de}$, so $K_{abc} = \omega_{ab} u_c$ is materially nonrotating. The unique orthogonally invariant materially nonrotating $\underset{\sim}{u}$-autoparallel contorsion which we determined as $K_{abc} = (u_{[a;b]} + u_{[a}\dot{u}_{b]})u_c$ has a covariant derivative associated with it which we denote by "$\dot{|}$" (to be distinguished from i). Then both the $\dot{|}$ and ! contorsions satisfy $K_{abc}u^c = u_{[b;a]} + u_{[b}\dot{u}_{a]}$, and both, of course, are materially nonrotating. Also ω_{ab} identifies the time rate $\frac{dQ}{d\tau}$ of the orthogonal transformation Q between spatially nonrotating and materially nonrotating frames, as shown below.

Let us compare the contorsion $K_{abc} = \omega_{ab} u_c$ with the zero contorsion. Then $\Delta K_{abc} = \omega_{ab} u_c$ so $\Lambda_{kj} = -\omega_{ab} r^a_{(k)} r^b_{(j)}$ and $\Lambda = Q \frac{dQ}{d\tau}$ where $\underset{\sim}{r}_{(1)}, \underset{\sim}{r}_{(2)}, \underset{\sim}{r}_{(3)}$ is the spatially nonrotating frame and $\bar{\underset{\sim}{r}}_{(1)}, \bar{\underset{\sim}{r}}_{(2)}, \bar{\underset{\sim}{r}}_{(3)}$ is the materially nonrotating frame with $\bar{\underset{\sim}{r}}_{(k)} = Q_k{}^j \underset{\sim}{r}_{(j)}$. Also we can easily see that $A_{kj} = -\omega_{ab} \bar{r}^a_{(k)} \bar{r}^b_{(j)}$ where $A = \left(\frac{dQ}{d\tau}\right) Q^T = Q \Lambda Q^T$.

We can, of course, let the bar frame be the materially nonrotating frame and the unbar frame be arbitrary. Then

$$\bar{W} = 0 = QWQ^T + A = Q(W + \Lambda)Q^T \Rightarrow W = -\Lambda = -Q^T \frac{dQ}{d\tau} .$$

This relates the 3-matrix W to the transformation $Q(\tau)$ and the rotation between frames.

(I.16) Vorticity and the Delta Tensors.

Recall that $\varepsilon_{abc} = \varepsilon_{abcd}u^d$ is orthogonal space time and completely skew. The unique vector field υ^c which is orthogonal ($\upsilon^c u_c = 0$) and satisfies $\omega_{ab} = \varepsilon_{abc}\upsilon^c$ is called the *vorticity.*

We can define $\delta^{ab}_{cd} = 2!\delta^{[a}_c\delta^{b]}_d$, $\delta^{abc}_{def} = 3!\delta^{[a}_d\delta^b_e\delta^{c]}_f$ and $\delta^{abcd}_{efgh} = 4!\delta^{[a}_e\delta^b_f\delta^c_g\delta^{d]}_h$. Clearly,

$$\delta^{abc}_{def} = \begin{vmatrix} \delta^a_d & \delta^a_e & \delta^a_f \\ \delta^b_d & \delta^b_e & \delta^b_f \\ \delta^c_d & \delta^c_e & \delta^c_f \end{vmatrix},$$

and $\delta^{abc}_{dec} = 2\delta^{ab}_{de}$, $\delta^{ab}_{db} = 3\delta^a_d$, $\delta^a_a = 4$ and the determinant result holds similarly for δ^{ab}_{cd} and δ^{abcd}_{efgh}. Of course we have $\varepsilon^{abcd}\varepsilon_{efgh} = -\epsilon^{abcd}\epsilon_{efgh} = -\delta^{abcd}_{efgh}$ and also, $\delta^{abcd}_{efch} = \delta^{abd}_{efh} = -\varepsilon^{abcd}\varepsilon_{efch}$. Hence,

$$\varepsilon^{abc}\varepsilon_{efc} = -\delta^{abd}_{efh}u^h u_d = \gamma^{ab}_{ef} \equiv 2\gamma^{[a}_e\gamma^{b]}_f = \gamma^a_e\gamma^b_f - \gamma^b_e\gamma^a_f$$

where $\gamma^a_e = \delta^a_e + u^a u_e$. Since $\omega_{ab} = \varepsilon_{abc}\upsilon^c$ we can write $\upsilon^a = \frac{1}{2}\varepsilon^{abc}\omega_{bc}$ as the direct solution for the vorticity. This can be seen immediately from the relation $\varepsilon^{abc}\varepsilon_{efc} = \gamma^{ab}_{ef}$.

(I.17) Covariant Differentiation and Curvature.

We can see that $A^a{}_{|bc} - A^a{}_{|cb} = -R^a{}_{dbc}A^d - A^a{}_{|d}T_b{}^d{}_c$ (Ricci identity) where $R^a{}_{dbc} = \Gamma_d{}^a{}_{c,b} - \Gamma_d{}^a{}_{b,c} + \Gamma_d{}^f{}_c\Gamma_f{}^a{}_b - \Gamma_d{}^f{}_b\Gamma_f{}^a{}_c$ is called the *Riemann tensor* and $T_b{}^d{}_c = \Gamma_b{}^d{}_c - \Gamma_c{}^d{}_b = -2K_{[b}{}^d{}_{c]}$ is called the *Torsion tensor.* For a metric connection the torsion determines the contorsion as $K_{cdb} = \frac{1}{2}(T_{bdc} + T_{dbc} - T_{bcd})$. We say a connection is *symmetric* if the torsion is zero, so the only symmetric metric connection is the Christoffel symbol connection. We define the contorsion curvature tensor by

$$C^a{}_{dbc} = K_d{}^a{}_{c;b} - K_d{}^a{}_{b;c} + K_d{}^f{}_b K_f{}^a{}_c - K_d{}^f{}_c K_f{}^a{}_b \,, \tag{I.17.1}$$

$$= -K_d{}^a{}_{[b;c]} - K_d{}^a{}_{[b|c]} + \frac{1}{2} K_d{}^a{}_f T_c{}^f{}_b \,,$$

so that $R^a{}_{dbc} = \mathcal{R}^a{}_{dbc} - C^a{}_{dbc}$ where $\mathcal{R}^a{}_{dbc}$ is the Christoffel symbol Riemann tensor, i.e.

$$\mathcal{R}^a{}_{dbc} = \left\{{a \atop d\ c}\right\}_{,b} - \left\{{a \atop d\ b}\right\}_{,c} + \left\{{f \atop d\ c}\right\}\left\{{a \atop f\ b}\right\} - \left\{{f \atop d\ b}\right\}\left\{{a \atop f\ c}\right\}.$$

For a metric contorsion it is clear that $R_{adbc} = R_{[ad][bc]}$, and $C_{adbc} = C_{[ad][bc]}$. If the covariant derivative $|$ also satisfies $u_{a|b} = 0$ then $R_{adbc}u^a = 0 = R_{adbc}u^d$.

(I.18) Flow-Constant Contorsions.

A metric covariant derivative is *flow constant* if $u_{a|b} = 0$ (and hence $u^a{}_{|b} = 0$). For example, the dot contorsion[1] $\dot{K}_{acb} = u_c u_{a;b} - u_a u_{c;b}$ gives rise to a flow constant derivative $(u_{a\cdot b} = 0)$ called the dot covariant derivative. It is spatially nonrotating and satisfies $\gamma_{ab\cdot c} = 0$, $\varepsilon_{abc\cdot d} = 0$, $\varepsilon^{abc}{}_{\cdot d} = 0$ etc.. There are also materially nonrotating flow constant contorsions, the simplest one being the $*$ contorsion $\overset{*}{K}_{acb} = u_{a;b}u_c - u_{c;b}u_a + (u_{[a;c]} - u_{[a}\dot{u}_{c]})u_b = 2u_{[c}u_{a];b} + \omega_{ac}u_b$. As above we have $u_{a*b} = 0$, $g_{ab*c} = 0$, $\varepsilon_{abc*d} = 0$, $\gamma_{ab*c} = 0$ etc. The nonuniqueness indicates the need to introduce material uniformity on the Body manifold Wang [108] p. 40 and extend the notion of a material connection on a body manifold [108, p. 62] to a material connection on the space time.

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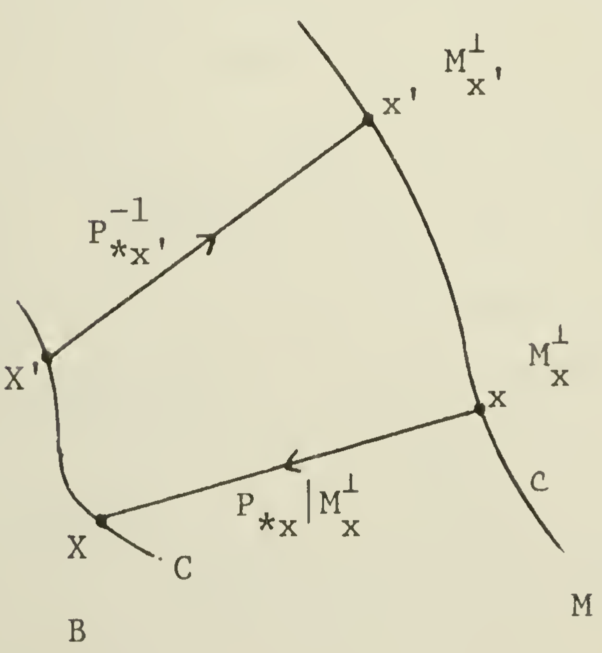
[1] The dot "•" does not indicate a time derivative in this case.

(I.19) <u>The Material Derivative</u>.

Let us suppose that a material uniformity is defined on the body manifold B, and a material connection is given on B. See Wang [108] p. 41, 62 and preliminaries (P.8) of this thesis for further details. Then in this section we show that the connection on the body manifold can be lifted naturally to a non-metric connection on M which we call the material connection.

For each tensor field $T^{ab\cdots}{}_{cd\cdots}$ on M we introduce the notion of the material derivative of T denoted by $T^{ab\cdots}{}_{cd\cdots\wedge e}$ a tensor field of one higher covariant rank like the usual covariant derivative. It will have the property that $T^{ab\cdots}{}_{cd\cdots\wedge e}u^e = \mathcal{D}T^{ab\cdots}{}_{cd\cdots}$ and $u^a{}_{\wedge e} = 0 = u_{a\wedge e}$, but $g_{ab\wedge c} = \gamma_{ab\wedge c} \neq 0$ so the material contorsion $\hat{K}_{abc}$ is not antisymmetric, i.e. $\hat{K}_{abc} \neq \hat{K}_{[ab]c}$. Of course $T^a{}_{\wedge e} = T^a{}_{;e} - T^b\hat{K}_b{}^a{}_e$ and $T_{a\wedge e} = T_{a;e} + T_b\hat{K}_a{}^b{}_e$ so that $T^a{}_{\wedge e}u^e = \mathcal{D}T^a \Rightarrow \hat{K}_{bae}u^e = u_{a;b} + u_a\dot{u}_b$ and $T_{a\wedge e}u^e = \mathcal{D}T_a \Rightarrow \hat{K}_{abe}u^e = u_{b;a} + u_b\dot{u}_a$. Also $u^a{}_{\wedge e} = 0 \Rightarrow \hat{K}_{bae}u^b = u_{a;e}$, $u_{a\wedge e} = 0 \Rightarrow \hat{K}_{abe}u^b = -u_{a;e}$. These are consistent, giving $\hat{K}_{abe}u^e u^b = -\dot{u}_a$ and $\hat{K}_{bae}u^b u^e = \dot{u}_a$.

To define the material derivative (which represents the deviation of a tensor field from material transport) let us first define material transport. If $\mathcal{C}$ is an arbitrary curve in M and $x \in \mathcal{C}$ and $w^a = \lambda u^a + v^a$ is a vector at x (in component form) we seek to transport it to $x' \in \mathcal{C}$, where $v^a|_x \in M^{\perp}_x$. We write the transported vector as $w^a|_{x'} = \lambda u^a|_{x'} + v^a|_{x'}$ where $v^a|_{x'} \in M^{\perp}_{x'}$ is the image under $P^{-1}_{*x'}$ of the transport from X to X' along the image



curve C in the body manifold B of $\mathcal{C}$ under P. The initial vector at X is $P_{*X}(w^a|_x)$ and it is transported along C in B using the given material connection on the body manifold (P.8). In particular the flow vector is preserved by material transport and in the generalization to tensors we shall see that the material transport of an orthogonal tensor field along any curve will remain orthogonal.

Let w^a be a vector field defined on a curve $\mathcal{C}$ which is parametrized by a parameter σ. We construct the parametric material derivative $\frac{\hat{\delta} w^a}{d\sigma}$ as follows. Let $m\left(w^a(x')\right)$ be the vector at $x \in \mathcal{C}$ (in M_x) obtained by materially transporting $w^a(x') \in M_{x'}$ along $\mathcal{C}$ to x where $x' \in \mathcal{C}$. Let $\Delta\sigma$ be the parameter increment in going from x to x'. Then $\frac{\hat{\delta} w^a}{d\sigma} = \lim\limits_{\Delta\sigma\to 0} \frac{m\left(w^a(x')\right) - w^a(x)}{\Delta\sigma}$. If w^a is defined on an open neighborhood $U \subset M$ and s^a is another vector field defined on U and if $x \in U$, $\mathcal{C}$ is the integral curve of s^a passing through x with natural parameter σ, then the material derivative $w^a{}_{\wedge b}$ of w^a at x is defined so that $w^a{}_{\wedge b} s^b = \frac{\hat{\delta} w^a}{d\sigma}$. This defines material differentiation of vector fields, by requiring this for all fields s^a. The Leibniz product rule and the condition $\phi_{\wedge b} = \phi_{,b}$ for scalars, defines material differentiation in general.

In particular, if we take any material tensor field (Wang [108] p. 44), "intrinsic" in his notation, it has zero covariant derivative using a material connection *ibid.* p. 74 and hence its lift to M will have zero material derivative. Therefore $\hat{n}^*_{abc\wedge d} = 0$ and $\hat{n}^{abc}{}_{\wedge d} = 0$.

Suppose $w^c u_c = 0$ at x and $v^c u_c = 0$ along a length of curve through x whose tangent is w^c at x. Then

$$\begin{aligned} v^a{}_{\wedge b} w^b &= v^a{}_{;b} w^b - v^c \hat{K}_c{}^a{}_b w^b \\ &= P^a_{*\alpha} v^\alpha{}_{|\beta} w^\beta \quad \text{at } x \in M. \end{aligned}$$

Here v^α is the P_* projection of v^a and w^β is the P_* projection of w^b $(a,b = 1,2,3,4, \alpha,\beta = 1,2,3)$, $|$ denotes covariant differentiation on B and we have used hypersurface orthogonal projection (see note to follow in the next section). Of course $v^a{}_{\wedge b}w^b$ is orthogonal to u_a and we have $v^\alpha = P_{*a}^{\ \alpha}v^a$, $v^a = P_{*\alpha}^{\ a}v^\alpha$, $w^\beta = P_{*b}^{\ \beta}w^b$ and $w^b = P_{*\beta}^{\ b}w^\beta$ with $P_{*\alpha}^{\ a}P_{*b}^{\ \alpha} = \gamma^a_b$, $P_{*\alpha}^{\ a}P_{*a}^{\ \beta} = \delta^\beta_\alpha$. In fact,

$$P_{*a}^{\ \alpha} = \text{projection} : M^\perp_x \to B_X, \quad P_{*\alpha}^{\ a} = \text{inverse} : B_X \to M^\perp_x \quad \text{and}$$
$$P^{*\alpha}_{\ a} = \text{projection} : \overset{*}{B}_X \to M^{*\perp}_x, \quad P^{*a}_{\ \alpha} = \text{inverse} : M^{*\perp}_x \to \overset{*}{B}_X$$

where $P^{*\alpha}_{\ a}P^{*b}_{\ \alpha} = \gamma^b_a$ and $P^{*\alpha}_{\ a}P^{*a}_{\ \beta} = \delta^\alpha_\beta$. In component form $P_{*a}^{\ \alpha} = P^{*\alpha}_{\ a}$ and $P_{*\alpha}^{\ a} = P^{*a}_{\ \alpha}$ so we can denote these transformations (or mixed tensors) by P^α_a and P^a_α respectively. Clearly $P^\alpha_a u^a = 0$ and $P^a_\alpha u_a = 0$.

Using this we obtain the formula for the material derivative as

$$v^a{}_{\wedge b} = P^a_\alpha P^\beta_b (v^d P^\alpha_d)_{|\beta} - u_b \mathcal{D}v^a - u^a(u_c v^c)_{,d}\gamma^d_b \ ,$$

and

$$v_{a\wedge b} = P^\alpha_a P^\beta_b (v_d P^d_\alpha)_{|\beta} - u_b \mathcal{D}v_a - u_a(u^c v_c)_{,d}\gamma^d_b \ .$$

To see this check $u_a v^a{}_{\wedge b}$, $v^a{}_{\wedge b}u^b$ and read the following section (I.20). For higher order tensors the results are similar. Notice that $g_{\alpha\beta}(\tau) = g_{ab}P^a_\alpha P^b_\beta = \gamma_{ab}P^a_\alpha P^b_\beta$ depends on the point $x \in P^{-1}(X)$ under consideration. Notice that raising and lowering of indices does not commute with material differentiation.

(I.20) <u>Note on Covariant Derivatives and Hypersurface Projections</u>.

In the expression for $v^a{}_{\wedge b}$ we have to determine $(v^d P^\alpha_d)_{|\beta} = v^\alpha{}_{|\beta}$. Since v^a may project to a different v^α for various points $x \in P^{-1}(X)$ we have to make the meaning of v^α and $v^\alpha{}_{|\beta}$ clear. Actually we choose a hypersurface H in M containing x which is orthogonal to u^a at x. (We cannot in general obtain a hypersurface H orthogonal to u^a over a

neighborhood of x in H). We then use P_* to transform this vector field v^a on H to v^α on a neighborhood of X in B. v^α depends on the choice of H in general on $P(H)$, but v^α and $v^\alpha{}_{|\beta}$ at x are independent of H. In fact, not only is the metric $g_{\alpha\beta}(\tau)$ function of proper time defined at X, but $g_{\alpha\beta,\gamma}(\tau)$ is well defined and independent of the choice of the hypersurface H at x orthogonal to $\underset{\sim}{u}$ at x. Hence we can define the Christoffel symbols

$$\left\{ {\alpha \atop \beta\ \gamma} \right\}(\tau) = \frac{1}{2}\, g^{\alpha\delta}(\tau)\left(g_{\beta\delta,\gamma} + g_{\gamma\delta,\beta} - g_{\beta\gamma,\delta}\right)(\tau)$$

at X and the contorsion $K_\alpha{}^\gamma{}_\beta(\tau)$ defined so that $\Gamma_\alpha{}^\gamma{}_\beta = \left\{ {\gamma \atop \alpha\ \beta} \right\}(\tau) - K_\alpha{}^\gamma{}_\beta(\tau)$ is the τ-independent material connection on B.

(I.21) The Fundamental Contorsion.

The material connection is $\hat{\Gamma}_a{}^b{}_c = \left\{ {b \atop a\ c} \right\} - \hat{K}_a{}^b{}_c$ on M where $\hat{K}_a{}^b{}_c$ is the material contorsion. We define $K_{abc} = \hat{K}_{[ab]c} = \frac{1}{2}(\hat{K}_{abc} - \hat{K}_{bac})$ and call it the *fundamental contorsion* with $\Gamma_a{}^b{}_c = \left\{ {b \atop a\ c} \right\} - K_a{}^b{}_c$ the fundamental connection with associated fundamental covariant derivative denoted by $|$ from now on unless otherwise specified. It is easy to see that $K_{abc}u^c = u_{[b;a]} + u_{[b}\dot{u}_{a]}$ so $|$ defines frames which rotate with the material medium along flow lines. Also we can see that $|$ is flow constant, i.e. $u_{a|b} = 0$ and $u^a{}_{|b} = 0$, so $g_{ab|c} = 0 \Rightarrow \gamma_{ab|c} = 0$, $\varepsilon_{abc|d} = 0$ etc. We say $\Gamma_a{}^b{}_c$ is the fundamental connection obtained by lifting the material connection $\Gamma_\alpha{}^\beta{}_\gamma$ from B to M, as $\hat{\Gamma}_a{}^b{}_c$ is the material connection obtained by lifting $\Gamma_\alpha{}^\beta{}_\gamma$ to M. It is easy to see that $v^a{}_{|b} = \dfrac{v^a{}_{\wedge b} + v^a{}_{\uparrow\wedge b}}{2}$ and $v_{a|b} = \dfrac{v_{\downarrow a\wedge b} + v_{a\wedge b}}{2}$ for any vector field v^a, where $v^a{}_{\uparrow\wedge b} = g^{ac}v_{c\wedge b}$.

The material contorsion K_{abc} must satisfy

$$\hat{K}_{abc}u^c = u_{b;a} + u_b\dot{u}_a, \quad \hat{K}_{abc}u^b = -u_{a;c}, \quad \hat{K}_{abc}u^a = u_{b;c} .$$

The simplest solution is

$$\hat{K}_{abc} = u_{a;c}u_b - u_{b;c}u_a - u_{b;a}u_c - \dot{u}_b u_a u_c . \tag{I.21.1}$$

The most general solution is obtained by adding an orthogonal space time covariant tensor field of order three to the above.

If we calculate $\hat{K}_{[ab]c} = K_{abc}$ for the simple solution above we obtain the contorsion for the $*$ covariant differentiation which we defined earlier. Evidently the most general fundamental contorsion is obtained by taking the $*$ contorsion and adding a tensor field $\Delta_{abc} = \Delta_{[ab]c}$ which is orthogonal.

(I.22) <u>The Delta Tensor</u>.

We use the delta tensor to indicate the orthogonal part of the contorsion $(\Delta_{abc} = \hat{\Delta}_{[ab]c})$ so,

$$\begin{aligned} \hat{K}_{abc} &= \hat{\Delta}_{abc} + u_{a;c}u_b - u_{b;c}u_a - u_{b;a}u_c - \dot{u}_b u_a u_c , \\ K_{abc} &= \Delta_{abc} + u_{a;c}u_b - u_{b;c}u_a - u_c(u_{[b;a]} + \dot{u}_{[b}u_{a]}) . \end{aligned} \tag{I.22.1}$$

We can write $v_{a\wedge b} = v_{a;b} + v_c\hat{K}^c{}_{a\,b}$ and compare with the component form of $v_{a\wedge b}$ we have, to obtain

$$P^\alpha_a P^\beta_b v_{\alpha|\beta} = \gamma^e_a \gamma^f_b v_{e;f} + \left(\hat{\Delta}^{\ c}_{a\ b} + (u_{a;b} + \dot{u}_a u_b)u^c\right)v_c .$$

If we introduce the metric $g_{\alpha\beta}(\tau)$ on B with Christoffel symbols $\left\{{}^{\alpha}_{\beta\ \gamma}\right\}(\tau)$ given as a function of τ and covariant differentiation ";" on the body, we may write (for v_c orthogonal) $P^\alpha_a P^\beta_b(v_{\alpha|\beta} - v_{\alpha;\beta}) = \hat{\Delta}^{\ c}_{a\ b}v_c$. Let $v_{\alpha|\beta} = v_{\alpha;\beta} + v_\gamma K_\alpha{}^\gamma{}_\beta$ where $K_\alpha{}^\gamma{}_\beta$ is a function of τ as well as $X \in B$. Then $P^\alpha_a P^\beta_b K_\alpha{}^\gamma{}_\beta v_\gamma = \hat{\Delta}^{\ c}_{a\ b}v_c$ or $\hat{\Delta}^{\ c}_{a\ b} = P^\alpha_a P^\beta_b P^c_\gamma K_\alpha{}^\gamma{}_\beta$. Thus the

delta tensor $\hat{\Delta}$ at x is the lift of the contorsion $K_{\alpha}{}^{\gamma}{}_{\beta}$ at (X,τ) or x. Of course we have $\Gamma_{\alpha}{}^{\gamma}{}_{\beta} = \left\{{}_{\alpha}{}^{\gamma}{}_{\beta}\right\}(\tau) - K_{\alpha}{}^{\gamma}{}_{\beta}(\tau)$ where the left hand side is independent of τ. Hence $\frac{d}{d\tau}\left\{{}_{\alpha}{}^{\gamma}{}_{\beta}\right\} = \frac{d}{d\tau} K_{\alpha}{}^{\gamma}{}_{\beta}$ and $\mathcal{D}\hat{\Delta}_{a}{}^{c}{}_{b} = P^{\alpha}_{a}P^{\beta}_{b}P^{c}_{\gamma}\left(\frac{d}{d\tau} K_{\alpha}{}^{\gamma}{}_{\beta}\right)$. We can use $g_{\alpha\beta}(\tau)$ to raise and lower indices at (X,τ) to get $\Delta_{acb} = \hat{\Delta}_{[ac]b} = P^{\alpha}_{a}P^{\beta}_{b}P^{\gamma}_{c}K_{[\alpha\gamma]\beta}$. Since $\mathcal{D}\hat{\Delta}_{a}{}^{c}{}_{b} = P^{\alpha}_{a}P^{\beta}_{b}P^{c}_{\gamma}\left(\frac{d}{d\tau}\left\{{}_{\alpha}{}^{\gamma}{}_{\beta}\right\}\right)$ is independent of the material connection on the body manifold, namely $\Gamma_{\alpha}{}^{\gamma}{}_{\beta}$ we see that specifying $\hat{\Delta}_{a}{}^{c}{}_{b}$ at one $x \in M$ determines it at all points of $P^{-1}\big(P(x)\big)$, i.e. on the world line through x, without further reference to $\Gamma_{\alpha}{}^{\gamma}{}_{\beta}$. On the other hand $\Gamma_{\alpha}{}^{\gamma}{}_{\beta}$ specifies $\hat{\Delta}_{a}{}^{c}{}_{b}$ at x for each $x \in M$ and hence gives us the relation in its value between different world lines. Thus we ought to be able to calculate $\mathcal{D}\hat{\Delta}_{a}{}^{c}{}_{b}$ and this is done in the next section.

(I.23) <u>Convective Derivative of the Delta Tensor</u>.

Let v_a be orthogonal and convective invariant. We may write $P^{\alpha}_{a}P^{\beta}_{b}v_{\alpha|\beta} = \gamma^{e}_{a}\gamma^{f}_{b}v_{e;f} + \hat{\Delta}_{a}{}^{c}{}_{b}v_{c}$. Since $v_{e:f} = v_{e;f} + v_{c}\overset{\vdots}{K}_{e}{}^{c}{}_{f}$ where $\overset{\vdots}{K}_{e}{}^{c}{}_{f} = (u_{e}\dot{u}^{c} - \dot{u}^{c}u_{e})u_{f}$ we have $\gamma^{e}_{a}\gamma^{f}_{b}v_{e;f} = \gamma^{e}_{a}\gamma^{f}_{b}v_{e:f}$. Taking convective derivatives we get $0 = \gamma^{e}_{a}\gamma^{f}_{b}\mathcal{D}(v_{e:f}) + v_{c}\mathcal{D}\hat{\Delta}_{a}{}^{c}{}_{b}$. Now, $0 = \mathcal{D}v_{e} = v_{e:g}u^{g} + v_{a}u^{a}{}_{:e}$ and $\mathcal{D}(v_{e:f}) = v_{e:fg}u^{g} + v_{a:f}u^{a}{}_{:e} + v_{e:a}u^{a}{}_{:f}$. Since $0 = (\mathcal{D}v_{e})_{:f} = v_{e:gf}u^{g} + v_{e:g}u^{g}{}_{:f} + v_{a:f}u^{a}{}_{:e} + v_{a}u^{a}{}_{:ef}$ we see that

$$\begin{aligned}\mathcal{D}(v_{e:f}) &= (v_{e:fg} - v_{e:gf})u^{g} - v_{a}u^{a}{}_{:ef} \\ &= -\overset{\vdots}{R}_{edfg}v^{d}u^{g} - v_{e:d}\overset{\vdots}{T}_{f}{}^{d}{}_{g}u^{g} - v_{a}u^{a}{}_{:ef}\ ,\end{aligned}$$

where $\overset{\vdots}{R}_{edfg}$ is the Riemann tensor and $\overset{\vdots}{T}_{f}{}^{d}{}_{g}$ is the torsion tensor associated with the colon covariant differentiation (I.17). Now

$$\begin{aligned}\gamma^{e}_{a}\gamma^{f}_{b}v_{e:d}\overset{\vdots}{T}_{f}{}^{d}{}_{g}u^{g} &= \gamma^{e}_{a}\gamma^{f}_{b}v_{e:d}(\overset{\vdots}{K}_{g}{}^{d}{}_{f} - \overset{\vdots}{K}_{f}{}^{d}{}_{g})u^{g} \\ &= -\gamma^{e}_{a}\gamma^{f}_{b}v_{e:d}u^{d}\dot{u}_{f} = \gamma^{e}_{a}\gamma^{f}_{b}v_{d}u^{d}{}_{:e}\dot{u}_{f} = v_{d}u^{d}{}_{:a}\dot{u}_{b}\ .\end{aligned}$$

Hence

$$\gamma^e_a\gamma^f_b\mathcal{D}(v_{e:f}) = \gamma^e_a\gamma^f_b\dot{R}^{:d}{}_{efg}v_d u^g - v_d u^d{}_{:a}\dot{u}_b - v_d u^d{}_{:ef}\gamma^e_a\gamma^f_b ,$$

and so therefore

$$\mathcal{D}\hat{\Delta}_a{}^c{}_b = -\gamma^e_a\gamma^f_b\gamma^c_d\dot{R}^{:d}{}_{efg}u^g + u^c{}_{:a}\dot{u}_b + \gamma^c_d u^d{}_{:ef}\gamma^e_a\gamma^f_b .$$

(I.24) Generalized Expansion and Rotation.

Let $\mathcal{C}$ be a curve in M, $x \in \mathcal{C}$ and $(v^a) \in M_x$ a vector. We Fermi transport using the fundamental connection (equivalent to parallel transport since $u^a{}_{|b} = 0 \Rightarrow u^a{}_{|b}u^b = 0$). Thus on $P^{-1}(P(x))$, v^a is defined, and if σ is a parameter for $\mathcal{C}$ with tangent vector $s^d = \frac{dx^d}{d\sigma}$ then $0 = \frac{\delta v^a}{d\sigma} = \frac{dv^a}{d\sigma} + v^c\Gamma_c{}^a{}_b s^b$ on $\mathcal{C}$. If v^a is defined on a neighborhood of x we can write $\frac{\delta v^a}{d\sigma} = v^a{}_{|b}s^b$. Assuming this we can see that the material derivative of v^a along $\mathcal{C}$ is

$$\frac{\hat{\delta} v^a}{d\sigma} = v^a{}_{\wedge b}s^b \quad \text{and} \quad \overset{\downarrow}{v}_{a\wedge b}s^b = -[\hat{K}_{(ca)b}s^b]v^c .$$

Because of the symmetry of $\hat{K}_{(ca)b}$ in c and a we have the familiar non-rotating condition we had earlier for Fermi transport along the flow lines, where we forced the convective derivative to transform the transport invariant vector field through a symmetric tensor. Thus the fundamental connection gives rise to parallel transports preserving the metric that rotates with the material medium even along arbitrary curves $\mathcal{C}$ other than the flow line curves whose tangents are u^a.

We define the *generalized stretching rate tensor* $\check{\theta}_{abc} = \hat{K}_{(ab)c}$. $\check{\theta}_{abc}s^c$ is a symmetric tensor in a and c which describes the σ-parameter rate of change of stretch along the vector direction s^c. Then $\theta_{ab} = \check{\theta}_{abc}u^c$ is the stretch rate along flow lines, and

$\hat{K}_{abc} = K_{abc} + \breve{\theta}_{abc}$ where the fundamental contorsion satisfies $K_{abc}u^c = -\omega_{ab} + 2\dot{u}_{[a}u_{b]}$. Define $\omega_{abc} = -K_{dec}\gamma^d_a\gamma^e_b$ to be the *generalized rotation tensor*. Then $\omega_{abc}u^c = \omega_{ab}$ and $\omega_{abc} = -\Delta_{abc} - \omega_{ab}u_c$, with $K_{abc} = -\omega_{abc} + u_{a;c}u_b - u_{b;c}u_a$. Putting $\breve{\nu}^c{}_d = \frac{1}{2}\varepsilon^{cab}\omega_{abd}$ we call $\breve{\nu}^c{}_d$ the *generalized vorticity* which satisfies $\breve{\nu}^c{}_d u^d = \nu^c$ (the vorticity defined earlier). Then $\breve{\nu}^c{}_d = -\frac{1}{2}\varepsilon^{cab}K_{abd} = -\nu^c u_d - \frac{1}{2}\varepsilon^{cab}\Delta_{abd}$ so we can write the Orthogonalized Vorticity as $\nu^c{}_d = \breve{\nu}^c{}_e\gamma^e_d = -\frac{1}{2}\varepsilon^{cab}\Delta_{abd}$. Likewise we may orthogonalize out the components of the generalized stretch and write

$$\tilde{\theta}_{abc} = \gamma^e_a\gamma^f_b\breve{\theta}_{efc} \quad \text{and} \quad \theta_{abc} = \gamma^e_a\gamma^f_b\gamma^g_c\breve{\theta}_{efg} = \gamma^g_c\tilde{\theta}_{abg} .$$

Then we have $\tilde{\theta}_{abc}u^c = \theta_{ab} = \breve{\theta}_{abc}u^c$ as before. In fact

$$\begin{aligned}\breve{\theta}_{abc} &= \hat{K}_{(ab)c} = \hat{\Delta}_{(ab)c} - u_{(b;a)}u_c - \dot{u}_{(b}u_{a)}u_c \\ &= \theta_{abc} - \theta_{ba}u_c = \theta_{abc} - \theta_{ab}u_c \quad \text{where clearly } \hat{\Delta}_{(ab)c} = \theta_{abc}.\end{aligned}$$

Since θ_{ab} is orthogonal, $\breve{\theta}_{abc} = \tilde{\theta}_{abc}$ so we can drop all further reference to $\tilde{\theta}_{abc}$. Hence we obtain $\hat{\Delta}_a{}^c{}_b = \theta_a{}^c{}_b + \Delta_a{}^c{}_b$.

If we look for a covariant derivative Δ with contorsion $\overset{\Delta}{K}$ that satisfies

$$\gamma^e_a\gamma^f_b v_{e;f} + (u_{a;b} + \dot{u}_a u_b)u^c v_c = v_{e\Delta f}\gamma^e_a\gamma^f_b$$

and hence also

$$P^\alpha_a P^\beta_b v_{\alpha|\beta} = \gamma^e_a\gamma^f_b v_{e\Delta f} + \hat{\Delta}_a{}^c{}_b v_c, \quad \text{we then have}$$

$\overset{\Delta}{K}_{ecf} = u_{e:f}u_c - u_e u_{c:f} + \Lambda_{ec}u_f$ where Λ is antisymmetric. If the contorsion is to define a rotation along flow lines with the material medium and the flows are to be u-autoparallel then it is uniquely determined and $\Lambda_{ec} = u_{[e;c]} + u_{[e}\dot{u}_{c]}$. In this case Δ is the $\ast$ covariant derivative we defined earlier, and in fact is flow constant.

If we look for a covariant derivative $\circ$ that satisfies $\gamma^e_a\gamma^f_b v_{e\circ f} = \gamma^e_a\gamma^f_b v_{e\Delta f} + \Delta_a{}^c{}_b v_c$ so that $P^\alpha_a P^\beta_b v_{\alpha|\beta} = \gamma^e_a\gamma^f_b v_{e\circ f} + \theta_a{}^c{}_b v_c$ then we have $\overset{\circ}{K}_{ecf} = u_{e:f}u_c - u_{c:f}u_e + \Delta_{ecf} + \Lambda_{ec}u_f$ for antisymmetric Λ. Choosing Λ the same as before so as to make the contorsion flow-materially nonrotating and $\underset{\sim}{u}$-autoparallel we find $\overset{\circ}{K}_{ecf} = K_{ecf}$ the fundamental contorsion uniquely, which is also flow constant by the way. Hence if $|$ is material covariant derivative on the body (identified with Greek indices) and is fundamental derivative on space time (Latin indices) we have

$$P^\alpha_a P^\beta_b v_{\alpha|\beta} = \gamma^e_a\gamma^f_b v_{e|f} + \theta_a{}^c{}_b v_c \,.$$

Since $\quad v_{e\wedge f} = v_{e|f} + v_c\theta_e{}^c{}_f \quad$ we have

$$P^\alpha_a P^\beta_b v_{\alpha|\beta} = \gamma^e_a\gamma^f_b v_{e\wedge f}$$ which also follows directly from the original formula for $v_{e\wedge f}$ in component form. Also $g_{ab\wedge c} = \gamma_{ab\wedge c} = 2\breve{\theta}_{abc}$ since $g_{ab\wedge c} = g_{ab|c} + g_{db}\breve{\theta}_a{}^d{}_c + g_{ad}\breve{\theta}_b{}^d{}_c$ and $g_{ab|c} = 0$ since the fundamental contorsion is metric. Now it is easy to see that

$$\begin{aligned}\mathcal{D}v_e &= v_{e\wedge f}u^f = v_{e|f}u^f + v_c\breve{\theta}_e{}^c{}_f u^f \\ &= v_{e|f}u^f + \theta_e{}^c v_c \,.\end{aligned}$$

For tensors,

$$\mathcal{D}T^{a\cdots}{}_{c\cdots} = T^{a\cdots}{}_{c\cdots|e}u^e - T^{e\cdots}{}_{c\cdots}\theta^a{}_e - \cdots + T^{a\cdots}{}_{e\cdots}\theta^e{}_c + \cdots,$$

and

$$\mathcal{R}T^{a\cdots}{}_{c\cdots} = T^{a\cdots}{}_{c\cdots|e}u^e = T^{a\cdots}{}_{c\cdots *e}u^e = T^{a\cdots}{}_{c\cdots !e}u^e = T^{a\cdots}{}_{c\cdots \dot{|} e}u^e$$

is called the *Rotational Flow derivative* which measures the deviation of a tensor field from a metric Fermi transport along world lines rotating with the material medium. The different covariant derivatives in the above could be replaced by any $\underset{\sim}{u}$-autoparallel flow-materially non-rotating one (whose contorsion satisfies $K_{eaf}u^f = -u_{[e;a]} - u_{[e}\dot{u}_{a]}$).

Similarly we can define the *Spatial flow derivative* $\mathcal{S}T^{a\ldots}{}_{c\ldots} = T^{a\ldots}{}_{c\ldots:e}u^e = T^{a\ldots}{}_{c\ldots\bullet e}u^e$ which measures the proper time deviation of the tensor $T^{a\ldots}{}_{c\ldots}$ from the Christoffel symbol (or semicolon) Fermi transport or equivalently the colon parallel transport along the world lines. Here we can use any covariant derivative (to replace $:$ or $\bullet$) that satisfies $K_{abc}u^c = -2u_{[a}\dot{u}_{b]}$ i.e. that is $\underset{\sim}{u}$-autoparallel and spatially nonrotating. We then have

$$\mathcal{R}T^{a\ldots}{}_{c\ldots} = \mathcal{S}T^{a\ldots}{}_{c\ldots} - T^{e\ldots}{}_{c\ldots}\omega^a{}_e - \cdots + T^{a\ldots}{}_{e\ldots}\omega^e{}_c + \cdots$$

relating the two flow derivatives.

(I.25) <u>The Riemann Tensor</u>.

Recall that $R^a{}_{dbc} = \mathcal{R}^a{}_{dbc} - \mathcal{C}^a{}_{dbc}$ where $\mathcal{R}^a{}_{dbc}$ is the Christoffel symbol Riemann tensor and

$$\mathcal{C}^a{}_{dbc} = K_d{}^a{}_{c;b} - K_d{}^a{}_{b;c} + K_d{}^f{}_b K_f{}^a{}_c - K_d{}^f{}_c K_f{}^a{}_b$$

is the contorsion curvature tensor (I.17).

For example, corresponding to the covariant differentiation $:$ we have

$$\mathcal{C}_{adbc} = 4u_{[d}\dot{u}_{a]}u_{[c;b]} + 4(u_{[d}\dot{u}_{a]})_{;[b}u_{c]}\ .$$

For the $\bullet$ covariant differentiation,

$$\dot{\mathcal{C}}_{adbc} = 4(u_{[a}u_{d];[c})_{;b]} + u_{d;b}u_{a;c} - u_{d;c}u_{a;b}.$$

For the $*$ covariant differentiation

$$\overset{*}{\mathcal{C}}_{adbc} = 4(u_{[a}u_{d];[c})_{;b]} + 2(\omega_{da}u_{[c})_{;b]} + u_{d;b}u_{a;c} - u_{d;c}u_{a;b}$$
$$+ 4u_{[a}\omega_{d]f}u^f{}_{;[c}u_{b]}\ .$$

For the fundamental covariant derivative $|$ we have

$$C_{adbc} = \Delta_{dac;b} - \Delta_{dab;c} + \Delta_d{}^f{}_b \Delta_{fac} - \Delta_d{}^f{}_c \Delta_{fab}$$
$$+ 4(u_{[a}u_{d];[c})_{;b]} + 2(\omega_{da}u_{[c})_{;b]} + 4u_{[a}\Delta_{d]f[b}u^f{}_{;c]}$$
$$+ 4\omega^f{}_{[a}\Delta_{d]f[b}u_{c]} + u_{d;b}u_{a;c} - u_{d;c}u_{a;b}$$
$$+ 4u^f{}_{;[c}u_{b]}\omega_{f[a}u_{d]} .$$

In each case we see that $C_{adbc} = C_{[ad][bc]}$, a result which follows from the corresponding antisymmetry properties of the Riemann tensors for the Christoffel symbols and for general metric preserving connections.

(I.26) Material Derivative of the Volume Element.

We can see that $\varepsilon_{abcd\wedge e} = \breve{\theta}_e \varepsilon_{abcd}$ and $\varepsilon_{abc\wedge e} = \breve{\theta}_e \varepsilon_{abc}$ where $\breve{\theta}_e = \breve{\theta}_a{}^a{}_e$. To prove this we write $\varepsilon_{abcd\wedge e} = \varepsilon_{abcd|e} + \varepsilon_{fbcd}\breve{\theta}_a{}^f{}_e + \varepsilon_{afcd}\breve{\theta}_b{}^f{}_e + \varepsilon_{abfd}\breve{\theta}_c{}^f{}_e + \varepsilon_{abcf}\breve{\theta}_d{}^f{}_e$ and set $a = 1$, $b = 2$, $c = 3$, $d = 4$. Also $\varepsilon^{abcd}{}_{\wedge e} = -\breve{\theta}_e \varepsilon^{abcd}$ and $\varepsilon^{abc}{}_{\wedge e} = -\breve{\theta}_e \varepsilon^{abc}$. By contracting with u^e we get the familiar formulas for the convective derivatives of each of the ε tensors, since $\breve{\theta}_e u^e = \theta = \theta^a{}_a$.

As we have seen, the lifts of the given volume element tensors on B have zero material derivative, i.e. $\hat{\eta}^{abc}{}_{\wedge e} = 0$ and $\hat{\eta}^*_{abc\wedge e} = 0$. This was true since $\eta^{\alpha\beta\gamma}$ and $\overset{*}{\eta}_{\alpha\beta\gamma}$ had zero covariant derivative on the body since they were material or intrinsic volume tensors. But $\varepsilon^{abc} = \rho\hat{\eta}^{abc}$ and $\varepsilon_{abc} = \frac{1}{\rho}\hat{\eta}^*_{abc}$, so taking material derivatives we find $\breve{\theta}_e = -(\ell n\ \rho)_{,e}$. Thus θ_e is the negative of the gradient of the logarithm of the density, and measures the material's expansion relative to its volume along any specified direction. We also have $\breve{\theta}_e = \breve{\theta}_a{}^a{}_e = \theta_a{}^a{}_e - \theta_a{}^a u_e = \theta_e - \theta u_e$ where $\theta_e = \theta_a{}^a{}_e$ is the contraction of the orthogonal part of $\breve{\theta}_a{}^b{}_c$, and is therefore orthogonal. Thus

$\theta_e = -\gamma_e^{\ f}(\ell n\,\rho)_{,f} = \theta_{a\ e}^{\ a}$ gives us a condition on θ_{abc} the orthogonal general stretch tensor.

(I.27) Mixed Tensors and Generalized Fermi Transport.

Let M be space time and $\underline{u}$ the unit time like future pointing flow vector field on M. Let $\tilde{\underline{u}}$ be another unit time like future pointing vector field and consider a curve $\mathcal{C}$ in M with parameter σ and tangent $\hat{u}^a = \frac{dx^a}{d\sigma}$ at each point of $\mathcal{C}$, and let r^a be a vector field defined on $\mathcal{C}$. We suppose that r^a is Fermi Transported along $\mathcal{C}$ using a connection $\Gamma_{b\ c}^{\ a}$ that is not necessarily metric, but is done with respect to the field $\tilde{u}^a$, i.e. $\frac{\delta r^a}{d\sigma} = \tilde{u}^a\left(\frac{\delta\tilde{u}_b}{d\sigma}\right) r^b = \tilde{u}^a\tilde{u}_{b|c}\hat{u}^c r^b$. This transport process preserves orthogonality to $\tilde{\underline{u}}$ at each point of $x \in \mathcal{C}$ and preserves $\tilde{u}$ itself. Let $\tilde{\gamma}^a_b = \delta^a_b + \tilde{u}^a\tilde{u}_b$, $\tilde{\gamma}_{ab} = g_{ab} + \tilde{u}_a\tilde{u}_b$ and put $\tilde{P}^\alpha_a = P^\alpha_b\tilde{\gamma}^b_a = P^\alpha_a + \tilde{u}^\alpha\tilde{u}_a$ where $\tilde{u}^\alpha = P^\alpha_a\tilde{u}^a$. Put

$\tilde{P}^b_\beta = P^b_\beta - \dfrac{u^b\tilde{u}_\beta}{\underline{u}\cdot\tilde{\underline{u}}}$ where $\tilde{u}_\beta = \tilde{u}_b P^b_\beta$ and $\underline{u}\cdot\tilde{\underline{u}} = u_a\tilde{u}^a < 0$ since they are both time like and future pointing. Then it is easily seen that $\tilde{P}^\alpha_a\tilde{P}^a_\beta = \delta^\alpha_\beta$ and $\tilde{P}^\alpha_a\tilde{P}^b_\alpha = \tilde{\gamma}^b_a$. If we put $r^\alpha = P^\alpha_a r^a$ and $\tilde{r}^\alpha = \tilde{P}^\alpha_a r^a$ then $\frac{d}{d\sigma}\tilde{r}^\alpha = \tilde{P}^\alpha_a\frac{dr^a}{d\sigma} + r^a\tilde{P}^\alpha_{a,c}\hat{u}^c$ where $\hat{u}^c$ need not be normalized or even time like. Now we impose a condition of material transport along $\mathcal{C}$ preserving vectors in $\tilde{M}^\perp_x$, $x \in \mathcal{C}$ where $\tilde{M}^\perp_x$ consists of those vectors in M_x orthogonal to $\tilde{\underline{u}}|_x$. The condition is $0 = \frac{\delta\tilde{r}^\alpha}{d\sigma} = \frac{d\tilde{r}^\alpha}{d\sigma} + \tilde{r}^\beta\Gamma_{\beta\ \gamma}^{\ \alpha}\hat{u}^\gamma$ on the image of $\mathcal{C}$ under P in B where $\hat{u}^\gamma = P^\gamma_c\hat{u}^c$. Therefore $\tilde{u}^a\tilde{u}_{b|c}\hat{u}^c r^b = \frac{dr^a}{d\sigma} + r^b\Gamma_{b\ c}^{\ a}\hat{u}^c$ and $0 = \tilde{P}^\alpha_a\frac{dr^a}{d\sigma} + r^a\tilde{P}^\alpha_{a,c}\hat{u}^c + \tilde{r}^\beta\Gamma_{\beta\ \gamma}^{\ \alpha}\hat{u}^\gamma$. As a result we obtain $0 = \tilde{P}^\alpha_a\tilde{u}^a\tilde{u}_{b|c}\hat{u}^c r^b = \tilde{P}^\alpha_a\frac{dr^a}{d\sigma} + \tilde{P}^\alpha_a r^b\Gamma_{b\ c}^{\ a}\hat{u}^c$ or $0 = r^b\hat{u}^c\left(-\tilde{P}^\alpha_{b,c} - \tilde{P}^\beta_b\Gamma_{\beta\ \gamma}^{\ \alpha}P^\gamma_c + \tilde{P}^\alpha_a\Gamma_{b\ c}^{\ a}\right)$. We define the *mixed covariant derivative* of $\tilde{P}^\alpha_b$ as

$$\tilde{P}^\alpha_{b|c} = \tilde{P}^\alpha_{b,c} + \tilde{P}^\beta_b\Gamma_{\beta\ \gamma}^{\ \alpha}P^\gamma_c - \tilde{P}^\alpha_a\Gamma_{b\ c}^{\ a},$$

where $\Gamma_{\beta\ \gamma}^{\ \alpha}$ is the given material connection on B and $\Gamma_{b\ c}^{\ a}$ is the connection on M. If the Fermi transport condition is to hold for all curves c in M (i.e. for all tangents $\hat{u}^a$) and for all transported vector fields r^a then we require that $\tilde{P}^{\alpha}_{b|c} = 0$. To completely determine the connection $\Gamma_{b\ c}^{\ a}$ on M we simply have to specify $\Gamma_{b\ c}^{\ a}\tilde{u}_a$ and we do this by requiring $\tilde{u}_{a|b} = 0$. The connection $\Gamma_{b\ c}^{\ a}$ we get we denote by $\tilde{\Gamma}_{b\ c}^{\ a}$ the contorsion by $\tilde{K}_{b\ c}^{\ a}$ and the covariant differentiation by ~, and call these the $\tilde{u}$-*material connection (contorsion, differentiation)* respectively. Thus the Fermi transport becomes the parallel transport $\frac{\tilde{\delta} r^a}{d\sigma} = 0$, because we have taken $\tilde{u}_{a\sim b} = 0$. We let $\tilde{P}^{\alpha}_{b;c}$ be the mixed covariant derivative using the material connection on B and the Christoffel symbol connection on M, i.e.

$$\tilde{P}^{\alpha}_{b;c} = \tilde{P}^{\alpha}_{b,c} + \tilde{P}^{\beta}_{b}\Gamma_{\beta\ \gamma}^{\ \alpha}P^{\gamma}_{c} - \tilde{P}^{\alpha}_{a}\left\{{a \atop b\ c}\right\},$$

so that using $\tilde{P}^{\alpha}_{b\sim c} = 0$ we find $\tilde{K}_{b\ c}^{\ a} = -\tilde{P}^{a}_{\alpha}P^{\alpha}_{b;c} + \tilde{u}^{a}\tilde{u}_{b;c}$. Therefore $\tilde{u}^{a}_{\sim c} = \tilde{u}^{a}_{;c} - \tilde{u}^{b}\tilde{K}_{b\ c}^{\ a}$ and $\tilde{K}_{b\ c}^{\ a}\tilde{u}^{b} = \tilde{u}^{b}_{;c}\tilde{P}^{a}_{\alpha}\tilde{P}^{\alpha}_{b} = \tilde{u}^{a}_{;c}$ so that $\tilde{u}^{a}_{\sim c} = 0$ also. In particular, if $\underline{\tilde{u}} = \underline{u}$ everywhere we get the usual material derivative so $\hat{K}_{b\ c}^{\ a} = -P^{a}_{\alpha}P^{\alpha}_{b;c} + u^{a}u_{b;c}$ and $P^{\alpha}_{b\wedge c} = 0$. Also the condition $\tilde{P}^{\alpha}_{b\sim c} = 0$ means that likewise $\tilde{P}^{b}_{\alpha\sim c} = 0$ since δ^{α}_{β} and $\tilde{\gamma}^{a}_{b}$ have zero $\tilde{u}$-material derivative, $\left(\tilde{P}^{b}_{\alpha\sim c} = \tilde{P}^{b}_{\alpha,c} + \tilde{P}^{a}_{\alpha}\Gamma_{a\ c}^{\ b} - \tilde{P}^{b}_{\beta}\Gamma_{\alpha\ \gamma}^{\ \beta}P^{\gamma}_{c}\right)$. In particular $P^{b}_{\alpha\wedge c} = 0$. The contorsions can be written out as

$$\tilde{K}_{b\ c}^{\ a} = -\tilde{P}^{a}_{\alpha}\tilde{P}^{\alpha}_{b,c} - \tilde{P}^{a}_{\alpha}\tilde{P}^{\beta}_{b}P^{\gamma}_{c}\Gamma_{\beta\ \gamma}^{\ \alpha} + \tilde{\gamma}^{a}_{d}\left\{{d \atop b\ c}\right\} + \tilde{u}^{a}\tilde{u}_{b;c}, \quad \text{and}$$

$$\hat{K}_{b\ c}^{\ a} = -P^{a}_{\alpha}P^{\alpha}_{b,c} - P^{a}_{\alpha}P^{\beta}_{b}P^{\gamma}_{c}\Gamma_{\beta\ \gamma}^{\ \alpha} + \gamma^{a}_{d}\left\{{d \atop b\ c}\right\} + u^{a}u_{b;c}.$$

Since $P^{\alpha}_{b,c} = \frac{\partial X^{\alpha}}{\partial x^b \partial x^c} = P^{\alpha}_{c,b}$ we see that the torsion for the material connection is $\hat{T}_{b\ c}^{\ a} = u^{a}(u_{c,b} - u_{b,c}) + P^{a}_{\alpha}P^{\beta}_{b}P^{\gamma}_{c}T_{\beta\ \gamma}^{\ \alpha}$, where $T_{\beta\ \gamma}^{\ \alpha} = \Gamma_{\beta\ \gamma}^{\ \alpha} - \Gamma_{\gamma\ \beta}^{\ \alpha}$. Expressing the material contorsion in terms of the delta tensor

(I.21) we have

$$\hat{\Delta}_a{}^c{}_b + P^c{}_\alpha P^\alpha{}_{a;b} = u^c{}_{;b}u_a + u^c{}_{;a}u_b + \dot{u}^c u_a u_b$$

and the right hand side is symmetric in a and b. Hence we see that if there is no torsion on the body manifold then the delta tensor is symmetric, in fact $\hat{\Delta}_{[a}{}^c{}_{b]} = \frac{1}{2} P^c{}_\gamma P^\alpha{}_a P^\beta{}_b T_\beta{}^\gamma{}_\alpha$. Contraction with u^b gives us $P^c{}_\alpha P^\alpha{}_{a;b}u^b = -u^c{}_{;a}$, and of course we can see directly that $P^c{}_\alpha u^a P^\alpha{}_{a;b} = -u^c{}_{;b}$.

Since $\tilde{P}^\alpha{}_d u^d = (\underline{u}\cdot\underline{\tilde{u}})\tilde{u}^\alpha$ we get $P^\alpha{}_b = \tilde{P}^\alpha{}_d \lambda^d{}_b$ where $\lambda^d{}_b = \delta^d{}_b - \dfrac{u^d \tilde{u}_b}{\underline{u}\cdot\underline{\tilde{u}}}$.

From this we can show that

$$\tilde{K}_a{}^d{}_b = \tilde{\gamma}^e{}_a \hat{K}_e{}^c{}_b \lambda^d{}_c - \tilde{u}_a \tilde{u}^d{}_{;b} - \frac{u^d \tilde{u}_{a;b}}{\underline{u}\cdot\underline{\tilde{u}}}, \quad \text{and}$$

$$\tilde{K}_a{}^d{}_b - \hat{K}_a{}^d{}_b = -\tilde{u}_a \tilde{u}^d{}_{\wedge b} - \frac{u^d \tilde{u}_{a\wedge b}}{\underline{u}\cdot\underline{\tilde{u}}} - \frac{\tilde{u}_a u^d (\tilde{u}^e \tilde{u}_{e\wedge b})}{\underline{u}\cdot\underline{\tilde{u}}}.$$

For the <u>spatially nonrotating</u> Fermi transport we have r^a is "$|$" and "$;$" Fermi transported along C so

$$\frac{\overline{\delta} r^a}{d\sigma} = \tilde{u}^a \tilde{u}_{b|c} \hat{u}^c r^b = \frac{dr^a}{d\sigma} + r^b \Gamma_b{}^a{}_c \hat{u}^c \quad \text{and}$$

$$\frac{\delta r^a}{d\sigma} = \tilde{u}^a \tilde{u}_{b;c} \hat{u}^c r^b = \frac{dr^a}{d\sigma} + r^b \left\{{}_b{}^a{}_c\right\} \hat{u}^c.$$

Subtracting we get $\tilde{u}^a \tilde{u}_d K_b{}^d{}_c \hat{u}^c r^b = -r^b K_b{}^a{}_c \hat{u}^c$ and if this is to hold for all $\hat{u}^c$ and all r^b orthogonal to $\tilde{u}^a$ we have $\tilde{\gamma}^a{}_d \tilde{\gamma}^b{}_e K_{abc} = 0$. In case we are dealing with flow frames $(\tilde{u}^a = u^a)$ then $\gamma^a{}_d \gamma^b{}_e K_{abc} = 0$ so $K_{abc} = u_b w_{ac} + u_a v_{bc}$ for arbitrary tensors w_{ac} and v_{bc}. If the contorsion is metric $(K_{abc} + K_{bac} = 0)$ then $v_{ac} = -w_{ac}$. If we impose the additional condition $u_{a|c} = 0$ then $u^a{}_{|c} = 0$ also and $|$ is merely the "$\cdot$" covariant derivative, which is shown here to be spatially nonrotating along arbitrary curves in space time. Of course

since $\underset{\sim}{u}$ has zero "$\cdot$" derivative we may use parallel transports instead of Fermi transports.

(I.28) The Exterior Derivative.

A k-*form* $v_{a_1,\cdots,a_k}$ is a completely skew covariant tensor field of order (or rank) k. We define the *exterior derivative* d of $\underset{\sim}{v}$ to be a k+1-form given by $dv_{a_1,\cdots,a_k}\big|_{a_{k+1}} = (k+1)v_{[a_1\cdots a_k,a_{k+1}]}$. For instance, for a scalar field ϕ, $d\phi\big|_a = \phi_{,a}$, for a covariant vector field v_a, $dv_a\big|_b = 2v_{[a,b]} = v_{a,b} - v_{b,a}$, for a 2-form $v_{ab} = -v_{ba}$, $dv_{ab}\big|_c = 3v_{[ab,c]} = v_{ab,c} + v_{bc,a} + v_{ca,b}$ and for a 3-form v_{abc}, $dv_{abc}\big|_d = 4v_{[abc,d]} = v_{abc,d} - v_{bcd,a} + v_{cda,b} - v_{dab,c}$, and similarly for higher orders. It is easy to see that $dv_{a_1\cdots a_k}\big|_{a_{k+1}}$ is a (k+1)-form and that the exterior derivative of an exterior derivative is zero. If v_{abc} is a 3-form and u^a a vector field, then $\mathcal{L}_{\underset{\sim}{u}} v_{abc} = u^d dv_{abc}\big|_d + d(v_{abd}u^d)\big|_c$ and more generally for k-forms. A k-form with zero exterior derivative is said to be *closed*, while an *exact* k-form is the derivative of a k-1 form. Every exact form is closed, and the converse is true in a local neighborhood.

We can apply these results to our material medium and see that $d\varepsilon_{abc}\big|_d = \theta\varepsilon_{abcd}$ since $\mathcal{D}\varepsilon_{abc} = \mathcal{L}_{\underset{\sim}{u}}\varepsilon_{abc}$ because ε_{abc} is orthogonal and covariant. Also $d\hat{\eta}^*_{abc}\big|_d = 0$ and $\hat{\eta}^*_{abc;d} = (\rho\varepsilon_{abc})_{;d} = -\check{\theta}_d\hat{\eta}^*_{abc}$. This shows that the exterior derivative behaves more like a material than a metric derivative. Of course $d\,u_a\big|_b = 2(\omega_{ab} - \dot{u}_{[a}u_{b]})$.

(I.29) Exterior Derivative Lifting Theorem.

If v_α *is a* 1-*form on* B *then* $v_a = P^\alpha_a v_\alpha$ *is an orthogonal convective invariant* 1-*form on* M. *Furthermore* $dv_a\big|_b = P^\alpha_a P^\beta_b dv_\alpha\big|_\beta$, *and similarly for* 2-*forms* $v_{\alpha\beta}$ *and* 3-*forms* $v_{\alpha\beta\gamma}$ *on* B.

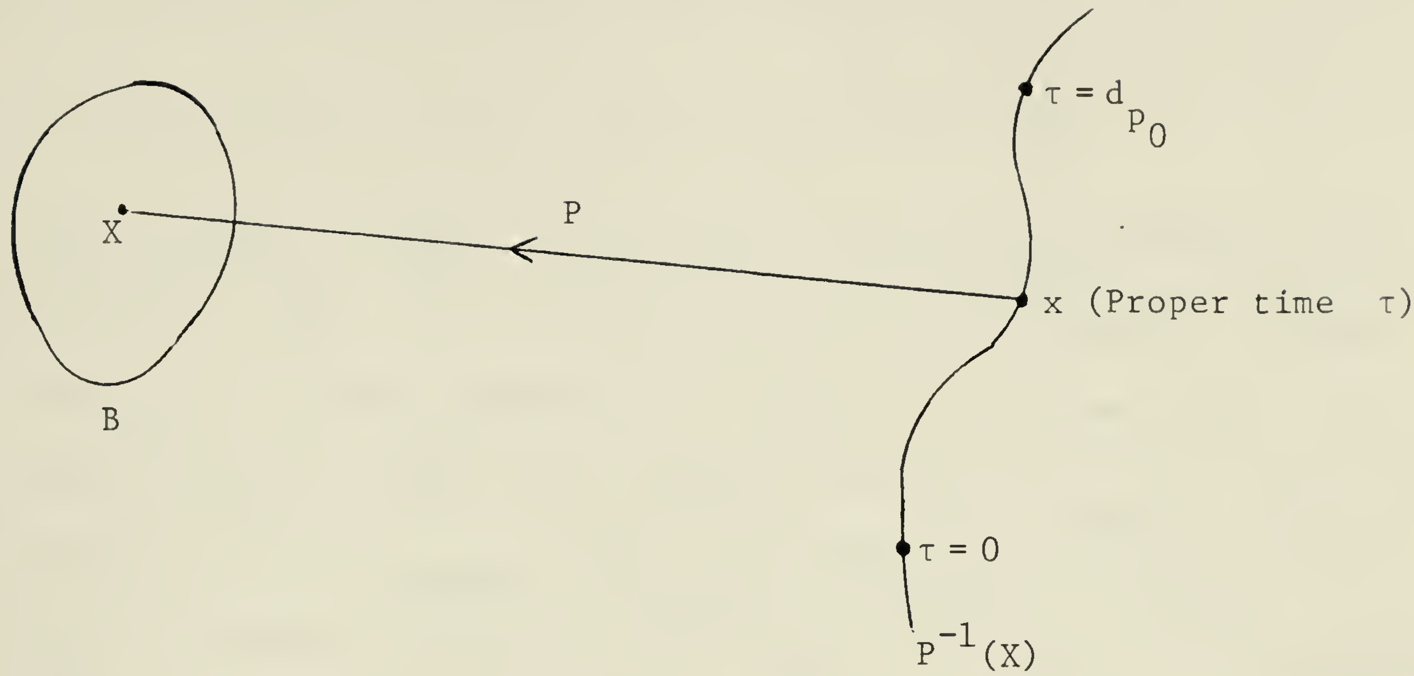
Proof: We show this for 1-forms only — the higher cases are similar but somewhat more complicated. Clearly $dv_{a|b}$ is orthogonal since $\mathcal{L}_u v_a = \mathcal{D}v_a = 0 = u^b\, dv_{a|b}$, in fact $dv_{a|b} = v_{a,b} - v_{b,a} = (P^\alpha_a v_\alpha)_{,b} - (P^\alpha_b v_\alpha)_{,a} = v_\alpha(P^\alpha_{a,b} - P^\alpha_{b,a}) + P^\alpha_a P^\beta_b(v_{\alpha,\beta} - v_{\beta,\alpha})$, so that $dv_{a|b} = P^\alpha_a P^\beta_b dv_{\alpha|\beta}$.

This result tells us that $d\hat{\eta}^*_{abc|d} = 0$ (as we have already seen) since $\eta^*_{\alpha\beta\gamma}$ is of maximal order and hence has zero exterior derivative on B. There is no projection theorem like this for non-orthogonal forms on M, for instance u_a has an exterior derivative with orthogonal part $2\omega_{ab}$, yet the projection of u_a to B is zero.

(I.30) Specific Motions - Motion of Constant Stretch History.

Certain conditions can be specified which describe the nature of the motion under consideration. If $\dot{u}^a = 0$ (on M or the region being examined) we say the motion is *geodesic* and if $\omega_{ab} = 0$ we say it is *irrotational*, and $du_{a|b} = 0$ if and only if the motion is both irrotational and geodesic. We will see later on that this is equivalent to the existence of symmetric Galilei connections (Chapter II). A motion is *isochoric* (volume preserving) if $\theta = 0$, *rigid* if $\theta_{ab} = 0$, and *shearfree* if $\sigma_{ab} = \theta_{ab} - \frac{\theta}{3}\gamma_{ab} = 0$. It is of *constant acceleration* if $\dot{u}^a$ is Christoffel Symbol Fermi invariant along flow lines, i.e. $\dot{u}^a{}_{;b}u^b = u^a(\dot{u}_b\dot{u}^b)$. It is worth extending at this time the notion of a motion of constant stretch history (see Truesdell [105] p. 65, Noll [77] p. 35) to a general relativistic setting.

Let us examine the deformation properties of a material medium in space-time. We consider the deformation process $P_0\colon [0,d_{p_0}] \to \mathrm{Sym}^+(B_X, B_X^*)$ determined by the motion of the material medium in the Lorentz space-time. Essentially $P_{(0)}(\tau) = \left(g_{\alpha\beta}(\tau)\right)$ [in components] is obtained by using the isomorphism $P_{*X}\colon M_X^\perp \to B_X$ to lower the metric determined



$\gamma_x \in \mathrm{Sym}^+(M_x^\perp, M_x^{\perp *})$ naturally related to the component form tensor $\gamma_{ab}(x) = g_{ab}(x) + u_a(x)u_b(x)$. If we let $p_x = P_{*x}$ then $P_{(0)}(\tau) = p_x^{-1*} \circ \gamma_x \circ p_x^{-1}$ for $\tau \in [0, d_{p_0}]$ where x is the point in $P^{-1}(X) \subset M$ associated with the world line proper time τ. In component form $P_{(0)\alpha\beta}(\tau) = (P^a_\alpha \gamma_{ab} P^b_\beta)(x)$, or alternatively $\gamma_{ab}(x) = P^\alpha_a P_{(0)\alpha\beta}(\tau) P^\beta_b$. Of course $P^\alpha_a = \dfrac{\partial X^\alpha}{\partial x^a}$ and $P^a_\alpha = \left.\dfrac{\partial x^a}{\partial X^\alpha}\right|_{\tau=\text{constant}}$. There is no unique way to relate the values of τ on adjacent world lines, however any relationship with the hypersurface τ = constant, orthogonal to the world line at the one point x defines P^a_α above at x. In general then, a different relationship is needed for different x.

We have seen that by placing material connections on both B and M that the material derivative of the mixed tensor P^α_a could be calculated and determined to equal zero, i.e.

$$0 = P^\alpha_{a \wedge b} = P^\alpha_{a,b} + P^\beta_a \Gamma^{\ \alpha}_{\beta\ \gamma} P^\gamma_b - P^\alpha_c \hat{\Gamma}^{\ c}_{a\ b} = P^\alpha_{a;b} + P^\alpha_c \hat{K}^{\ c}_{a\ b} .$$

In general, mixed covariant differentiation can be defined for vector fields on either M or B as below and extended using the Leibniz rule for differentiation of a product.

$$v^{\alpha}{}_{\wedge b} = v^{\alpha}{}_{,b} + v^{\beta}\Gamma_{\beta}{}^{\alpha}{}_{\gamma}P^{\gamma}_{b}, \quad w^{a}{}_{\wedge b} = w^{a}{}_{,b} + w^{c}\hat{\Gamma}_{c}{}^{a}{}_{b},$$

$$w_{a\wedge b} = w_{a,b} - w_{c}\hat{\Gamma}^{c}{}_{a\,b}, \quad v_{\alpha\wedge b} = v_{\alpha,b} - v_{\gamma}\Gamma^{\gamma}{}_{\alpha\,\beta}P^{\beta}_{b}.$$

Since $(P^{a}_{\alpha}P^{\alpha}_{b})_{\wedge c} = \gamma^{a}_{b\wedge c} = (\delta^{a}_{b} + u^{a}u_{b})_{\wedge c} = 0 = P^{a}_{\alpha\wedge c}P^{\alpha}_{b}$ (because $P^{\alpha}_{b\wedge c} = 0$) we see that $P^{a}_{\alpha\wedge c} = 0$. In mixed covariant differentiation, tensor fields on B may be allowed to be functions of proper time τ and not merely constant fields. For instance the field $P_{(0)\alpha\beta}(\tau)$ can be differentiated to get $P_{(0)\alpha\beta\wedge c} = P^{a}_{\alpha}\gamma_{ab\wedge c}P^{b}_{\beta} = P^{a}_{\alpha}g_{ab\wedge c}P^{b}_{\beta} = 2P^{a}_{\alpha}\breve{\theta}_{abc}P^{b}_{\beta}$. Of course $\breve{\theta}_{abc}$ is already orthogonal in its first two indices so the orthogonalizing effect of the projection can be ignored. Essentially this gives us the deformation expansion rate along the direction v^{c} when contracted with v^{c}. Of greatest interest is the deformation rate along flow lines, i.e.

$$\mathcal{D}P_{(0)\alpha\beta} = P_{(0)\alpha\beta\wedge c}u^{c} = 2P^{a}_{\alpha}\breve{\theta}_{abc}P^{b}_{\beta}u^{c} = 2P^{a}_{\alpha}\theta_{ab}P^{b}_{\beta}$$
$$= \frac{d}{d\tau}P_{(0)\alpha\beta} \quad \text{at fixed } X \in B.$$

Thus we have in component form $P_{(0)\alpha\beta} = P^{a}_{\alpha}\gamma_{ab}P^{b}_{\beta}$ and $\mathcal{D}P_{(0)\alpha\beta} = P^{a}_{\alpha}2\theta_{ab}P^{b}_{\beta}$. For a monotonous (constant stretch history or substantially stagnant) process P_0 we have (Noll [77] p. 35) $P_0(\tau) = \exp(\tau E^{*})P^{i}_{0}\exp(\tau E)$ for $E \in \mathrm{Lin}(B_X)$ where $P^{i}_{0} = P_0\ (\tau = 0)$. Thus $\mathcal{D}P_0(\tau) = \frac{d}{d\tau}P_0(\tau) = E^{*}P_0(\tau) + P_0(\tau)E$. Of course E is independent of τ for each X, but will depend upon $X \in B$ since $E \in \mathrm{Lin}(B_X)$. In component form on B we can write E as a tensor $E^{\alpha}{}_{\beta}$. If defined as a field on B, it can be lifted to an orthogonal tensor field on M which is materially invariant along flow lines and can be written as $E^{a}{}_{b}$. (Otherwise it is only defined on $P^{-1}(X)$). Hence $u^{b}E^{a}{}_{b} = 0 = u_{a}E^{a}{}_{b}$ and $\mathcal{D}E^{a}{}_{b} = 0$, $E^{a}{}_{b} = P^{a}_{\alpha}E^{\alpha}{}_{\beta}P^{\beta}_{b}$, and $E^{\alpha}{}_{\beta} = P^{\alpha}_{a}E^{a}{}_{b}P^{b}_{\beta}$. The mapping $E^{*} \in \mathrm{Lin}(B^{*}_{X})$ in components is given by $E^{*\beta}_{\alpha} = E^{\beta}{}_{\alpha}$. Hence $\frac{d}{d\tau}P_{(0)\alpha\beta}(\tau) = E^{*\gamma}_{\alpha}P_{(0)\gamma\beta} +$

$+ P_{(0)\alpha\gamma}E^{\gamma}{}_{\beta} = E^{\gamma}{}_{\alpha}P_{(0)\gamma\beta} + P_{(0)\alpha\gamma}E^{\gamma}{}_{\beta}$. Lifting this equation up to the space time we obtain,

$$P^{\alpha}_{a}P^{\beta}_{b}\mathcal{D}P_{(0)\alpha\beta} = P^{\alpha}_{a}P^{\beta}_{b}[E^{\gamma}{}_{\alpha}P_{(0)\delta\beta}P^{\delta}_{c}P^{c}_{\gamma} + P_{(0)\alpha\delta}E^{\gamma}{}_{\beta}P^{\delta}_{c}P^{c}_{\gamma}]$$

or

$$2\theta_{ab} = E^{c}{}_{a}\gamma_{cb} + \gamma_{ac}E^{c}{}_{b} = E_{ba} + E_{ab}$$

since $E^{c}{}_{a}$ is orthogonal. Therefore we obtain $E_{(ab)} = \theta_{ab}$. Hence the space time motion is monotonous if and only if there exists an antisymmetric orthogonal tensor W_{ab} such that if we define $E_{ab} = \theta_{ab} + W_{ab}$ then $\mathcal{D}E^{a}{}_{b} = 0$. We say the motion is a *viscometric flow* (as in Truesdell [105]) if $E^{a}{}_{b}E^{b}{}_{c} = 0$ identically. The tensor W_{ab} is called the *monotonous rotation tensor.*

We can introduce a metric contorsion $\overset{\circ}{K}_{cba}$ on M with $u^{a}{}_{\circ b} = E^{a}{}_{b} = \theta^{a}{}_{b} + W^{a}{}_{b}$ or equivalently $u_{a\circ b} = E_{ab}$. If we require the convective derivative to be expressed in Lie derivative form, then the general solution is

$$\overset{\circ}{K}_{cba} = -u_{a}(W_{cb} - \omega_{cb} + u_{b}\dot{u}_{c} - \dot{u}_{b}u_{c}) - u_{c}(W_{ab} - \omega_{ab}) - u_{b}(\omega_{ac} - W_{ac}) + \overset{\circ}{\Delta}_{cba} ,$$

where $\overset{\circ}{\Delta}_{cba}$ is orthogonal and satisfies $\overset{\circ}{\Delta}_{cba} + \overset{\circ}{\Delta}_{bca} = 0$. The monotonous flow condition is $\mathcal{D}E^{a}{}_{b} = 0 = \mathcal{D}u^{a}{}_{\circ b}$ which implies $u^{a}{}_{\circ bc}u^{c} = 0$, using the Lie derivative form for $\mathcal{D}$. If the motion is a viscometric flow, then $u^{a}{}_{\circ c}u^{c}{}_{\circ b} = 0$ and so $u^{a}{}_{\circ cb}u^{c} = 0$ since $u^{a}{}_{\circ c}u^{c} = E^{a}{}_{c}u^{c} = 0$. Therefore $(u^{a}{}_{\circ bc} - u^{a}{}_{\circ cb})u^{c} = 0$, and so since $\overset{\circ}{T}_{b}{}^{d}{}_{c}u^{c} = -u^{d}\dot{u}_{b}$ for the torsion, $\overset{\circ}{R}{}^{a}{}_{dbc}u^{d}u^{c} = 0$. Since $\overset{\circ}{\Gamma}{}_{a}{}^{b}{}_{c}$ is a metric connection, $-\overset{\circ}{R}_{adbc} = \overset{\circ}{R}_{dabc}$ and of course $\overset{\circ}{R}_{adbc} = -\overset{\circ}{R}_{adcb}$ as for all Riemann tensors. If any one index of $\overset{\circ}{R}_{adbc}$ is contracted with u^{e} the remaining covariant order 3 tensor is orthogonal (in the case of a

viscometric flow). For the torsion we have

$$-\overset{\circ}{T}{}_{c}{}^{b}{}_{a} = \overset{\circ}{K}{}_{c}{}^{b}{}_{a} - \overset{\circ}{K}{}_{a}{}^{b}{}_{c} = u^{b}(\dot{u}_{a}u_{c} - \dot{u}_{c}u_{a}) + 2u^{b}(\omega_{ca} - W_{ca}) + \overset{\circ}{\Delta}{}_{c}{}^{b}{}_{a} - \overset{\circ}{\Delta}{}_{a}{}^{b}{}_{c} \ .$$

We can use, (for a monotonous process) γ_{ca} to lower indices in $\mathcal{D}(\theta^{a}{}_{b} + W^{a}{}_{b}) = 0$ and from $\mathcal{D}\gamma_{ca} = 2\theta_{ca}$ we have, separating symmetric and antisymmetric parts,

$$\mathcal{D}W_{cb} = 2\theta_{a[c}W^{a}{}_{b]} \ ,$$

$$\mathcal{D}\theta_{cb} = 2\theta_{ca}\theta^{a}{}_{b} + 2\theta_{a(c}W^{a}{}_{b)} \ .$$

For a monotonous (or constant stretch history) flow $\mathcal{D}E^{a}{}_{b} = 0$ and hence $0 = \mathcal{D}E^{a}{}_{a} = \frac{d}{d\tau}\theta = \theta_{,a}u^{a}$, where $\theta = \theta^{a}{}_{a} = u^{a}{}_{;a}$. But $\theta = -\frac{d}{d\tau}\ell n\ \rho$ so $\ell n\ \rho = -\theta(X)\tau + \ell n \rho_{0}$ where $\rho_{0} = \rho(\tau = 0)$ on $P^{-1}(X)$, i.e. $\rho(x) = \rho_{0}e^{-\theta(X)\tau}$ where $x = \beta(\tau) \in P^{-1}(X)$. In isochoric motions $(\theta = 0)$, all deformations are shearings, a *shearing* being a motion for which $\sigma_{ab} = \theta_{ab}$ where by definition $\sigma_{ab} = \theta_{ab} - \frac{\theta}{3}\gamma_{ab}$.

CHAPTER II

GROUP STRUCTURES ON A MANIFOLD

(II.1) Definitions and Axioms.

Let M be an n-dimensional differentiable manifold (Dieudonné [22] p. 4). We will define what is meant by a group structure on M, a concept discussed by Kobayashi and Nomizu [44], Wang [108] and others. For each $x \in M$ we let M_x denote the tangent space to M at x which is an n-dimensional vector space. We let $GL(M_x)$, $GL^+(M_x)$, $Unim(M_x)$, $SL(M_x)$ denote the groups of all linear isomorphisms, those with positive determinant, those with determinant ± 1, and those with determinant $+1$ respectively.

A *Group Structure* on M is an association $x \to G(M_x) \subset GL(M_x)$ of a Lie subgroup $G(M_x)$ of $GL(M_x)$ with each $x \in M$ subject to axioms laid out below. We write $G_x = G(M_x)$. An isomorphism $\kappa: M_x \to M_y$ is *structure preserving* if $\kappa \circ G_x \circ \kappa^{-1} = G_y$. Trivially G_x is contained in the set of all structure preserving isomorphisms of M_x with itself.

Axiom I. *For each* $x,y \in M$ *there exists a structure preserving isomorphism* $\kappa: M_x \to M_y$ (Uniform nature of group structure). The map $\hat{\kappa}: G_x \to G_y$ defined by $\hat{\kappa}(g) = \kappa \circ g \circ \kappa^{-1}$ is a group isomorphism and a diffeomorphism, i.e. a Lie group isomorphism. Hence all the Lie groups G_x, $x \in M$ are isomorphic. We let $\mathcal{g}_x$ denote the Lie algebra of G_x which we will characterize later. Let V be an n-dimensional real vector space and let $G(V)$ be a Lie Subgroup of $GL(V)$ which is Lie group isomorphic to any (and every) G_x.

<u>Axiom II</u>. M *can be covered by a family of* <u>*reference charts*</u> [108, p. 40] *of the form* (U_α, r_α) where U_α is an open subset of M and $r_\alpha\colon T(U_\alpha) \to V$ is a smooth map $\left(T(U_\alpha) = \bigcup_{x \in U_\alpha} M_x\right)$ with $r_{\alpha x} = r_\alpha | M_x : M_x \to V$ a structure preserving isomorphism for each $x \in U_\alpha$. Thus $r_{\alpha x} \circ G_x \circ r_{\alpha x}^{-1} = G(V)$ and $\hat{r}_{\alpha x}$ is a Lie group isomorphism from G_x to $G(V) = G$. A maximal family $\{(U_\alpha, r_\alpha),\ \alpha \in A\}$ of reference charts is called a *reference atlas*. This axiom (Axiom II) describes the smooth nature of the group structure. Because compositions and inverses of structure preserving isomorphisms are structure preserving we see that Axiom I is implied by this axiom.

<u>Axiom III</u>. (Group sufficiency) *For all* $\alpha, \beta \in A$ *and all* $x \in U_\alpha \cap U_\beta$, $r_{\alpha x} \circ r_{\beta x}^{-1} \in G$. This implies from Axiom II above that $r_{\beta x}^{-1} \circ r_{\alpha x} \in G_x$.

The differential structure on a finite dimensional vector space such as V is taken to be the canonical one (Dieudonné [22] p. 7). We say that M is *locally homogeneous* (Wang [108] p. 50) if it can be covered by a family $\{(U_\alpha, r_\alpha),\ \alpha \in B\}$ of reference charts where $r_{\alpha x} = f_{\alpha * x}$, $x \in U_\alpha$ for some smooth embedding $f_\alpha\colon U_\alpha \to V$. We have made the natural identification of the tangent bundle TV with $V \times V$ here.

Let $\mathfrak{g}$ denote the Lie Algebra of $G = G(V)$. We can write the group structure on M in the following way. $\mathcal{G}(M) = (M, \mathcal{G}, \mathcal{U}, V, G)$ where $\mathcal{G} = \{G_x,\ x \in M\}$ and $\mathcal{U} = \{(U_\alpha, r_\alpha),\ \alpha \in A\}$. Since the reference atlas is maximal, $(U_\alpha, r_\alpha) \in \mathcal{U}$ if and only if $(U_\alpha, g \circ r_\alpha) \in \mathcal{U}$ for every $g \in G$. If $g_o \in GL(V)$ we can write $g_o\mathcal{U} = \{(U_\alpha, g_o \circ r_\alpha),\ \alpha \in A\}$ and we obtain another group structure $g_o\mathcal{G}(M) = (M, \mathcal{G}, g_o\mathcal{U}, V, g_o G g_o^{-1})$. The reference atlas $\mathcal{U}$ for any group structure is clearly unique.

Also $g_o G(M) = G(M)$ if and only if $g_o \in G$.

Let $V^1 = V$ and $V_1 = V^*$ the dual of V. More generally let V^k_ℓ denote the tensor space contravariant order k covariant order ℓ. Then $g \in G = G(V)$ induces a natural $g^* \in G^* = G(V^*)$ the dual map of the isomorphism $g: V \to V$. Likewise $g \in G$ induces $g^k_\ell \in G^k_\ell = G(V^k_\ell)$ and $g^k_\ell: V^k_\ell \to V^k_\ell$. A tensor $T \in V^k_\ell$ is said to be *invariant* if $g^k_\ell(T) = T$ for all $g^k_\ell \in G^k_\ell$. (G^k_ℓ consists of all isomorphisms of V^k_ℓ induced by elements of G). Every invariant tensor on V can be lifted uniquely to a smooth tensor field on M which is invariant at each point of M.

The tensor notation can include symmetric and skew tensors as well. For instance $V_{(2)}$ and $V_{[2]}$ are subspaces of $V_2 = V^0_2$ with $V_2 = V_{(2)} \oplus V_{[2]}$. $V_{(2)}$ consists of all symmetric covariant tensors of order 2 and $V_{[2]}$ all covariant skew tensors of order 2. In a similar way we define $V^{(3)}$, $V^{[4]}$, $V^1_{[2]}$, the latter being a torsion type of tensor. The existence of invariant tensors of various types depends completely on the group G and its properties. A tensor field on M which is a lift through the structure preserving isomorphisms of an invariant tensor on V is called an *invariant tensor field* on M.

There are a number of special types of group structures which are of particular interest. If M is an orientable differential manifold (i.e. is equipped with an oriented atlas of coordinate charts) and G is a subgroup of $GL^+(V)$ we say that $G(M)$ is an *orientation preserving* group structure. For any M, if G is a subgroup of $\mathrm{Unim}(V)$ we say that $G(M)$ is a *volume preserving* group structure. If both these conditions hold we have an orientation and volume preserving group structure, and G is a Lie subgroup of $SL(V)$.

The group structure on M defines for us some associated fibre bundles (Steenrod [101]). The *Lie tangent bundle* $\mathcal{T}(M,\mathcal{U}) = \left(T(M),M,P,V,G\right)$ associated with the group structure $\mathcal{G}(M)$ is defined naturally and uniquely in the following way. $T(M)$ is called the fibre space, M the base space, P the projection, V the fibre and G the structure group. We have $T(M) = \bigcup_{x\in M} M_x$ (disjoint union), $P\colon T(M) \to M$ defined by $P(M_x) = x$, $\forall\, x \in M$, and the bundle charts for $\mathcal{T}(M,\mathcal{U})$ are $\mathcal{U}_o = \{(U_\alpha,\phi_\alpha),\ \alpha \in A\}$ where $\mathcal{U} = \{(U_\alpha,r_\alpha),\ \alpha \in A\}$ and $\phi_\alpha\colon U_\alpha \times V \to T(U_\alpha) = \bigcup_{x\in U_\alpha} M_x$ is defined by $\phi_{\alpha x} = r_{\alpha x}^{-1}\colon V \to M_x$ for $x \in U_\alpha$, where $\phi_{\alpha x} = \phi_\alpha(x,\cdot)$. Each ϕ_α is a diffeomorphism and $\phi_{\alpha x}$, $x \in U_\alpha$ is a structure preserving isomorphism between V and M_x, and $\hat{\phi}_{\alpha x}\colon G \to G_x$ is a Lie group isomorphism. G acts naturally as a left translation on V (being a subgroup of $GL(V)$) and the coordinate transformations $G_{\alpha\beta}(x)\colon V \to V$ for $x \in U_\alpha \cap U_\beta$ defined by $G_{\alpha\beta}(x) = \phi_{\alpha x}^{-1} \circ \phi_{\beta x}$ are smooth maps $G_{\alpha\beta}\colon U_\alpha \cap U_\beta \to G$ where G has the Lie group differentiable structure.

There is one more bundle of importance worth considering here namely the *Lie principal bundle* $\mathcal{E}(M,\mathcal{U}) = \left(E(M,\mathcal{U}),M,\pi,G,G\right)$ associated with the group structure $\mathcal{G}(M) = (M,\mathcal{G},\mathcal{U},V,G)$. Here $E(M,\mathcal{U})$ is the fibre space, M is the base space, π is the projection, and G is both the fibre and the structure group. We let $L_x(\mathcal{U}_o) = \{\phi_{\alpha x}\colon V \to M_x \mid x \in U_\alpha,\ \alpha \in A\}$ and define $E(M,\mathcal{U}) = \bigcup_{x\in M} L_x(\mathcal{U}_o)$. The map $\pi\colon E(M,\mathcal{U}) \to M$ is given by $\pi\left(L_x(\mathcal{U}_o)\right) = x$. The bundle charts for $\mathcal{E}(M,\mathcal{U})$ are the collection $\mathcal{U}_1 = \{(U_\alpha,\psi_\alpha),\ \alpha \in A\}$ where $\psi_\alpha\colon U_\alpha \times G \to E(U_\alpha,\mathcal{U}) = \bigcup_{x\in U_\alpha} L_x(\mathcal{U}_o)$ is defined in terms of $\psi_{\alpha x}\colon G \to L_x(\mathcal{U}_o)$ by $\psi_{\alpha x}(g)(v) = \phi_{\alpha x}(gv) \in M_x$ for $g \in G \subset GL(V)$ and $v \in V$. The structure group G acts as a left translation on itself, and the coordinate transformations $\overline{G}_{\alpha\beta}(x)\colon G \to G$ defined by $\overline{G}_{\alpha\beta}(x) = \psi_{\alpha x}^{-1} \circ \psi_{\beta x}$ for $x \in U_\alpha \cap U_\beta$ correspond to left

translation by an element of G which turns out to be equal to $G_{\alpha\beta}(x)$. In a sense then, we can look upon the coordinate transforms $G_{\alpha\beta}(x)$ and $\overline{G}_{\alpha\beta}(x)$ as being the same.

We let $\mathrm{Lin}(V)$ denote the set of all linear maps from V to itself, and $GL(V)$ all the invertible ones. $\mathrm{Lin}(V)$ is a vector space under natural addition and scalar multiplication of maps, and it has dimension n^2 since V has dimension n. There is a natural map $\mathrm{Exp}: \mathrm{Lin}(V) \to GL(V)$ defined by $\mathrm{Exp}(a) = 1_V + \sum_{n=1}^{\infty} \frac{a^n}{n!}$ where $a \in \mathrm{Lin}(V)$, 1_V is the identity on V and $a^n = a \circ a \circ \cdots \circ a$ (n times). If G is a Lie subgroup of $GL(V)$ then the Lie algebra $\mathcal{g}$ of G consists of all those $a \in \mathrm{Lin}(V)$ for which $\mathrm{Exp}(ta) \in G$ for all $t \in \mathbb{R}$, i.e. $\mathcal{g} = \{a \in \mathrm{Lin}(V) \mid \mathrm{Exp}(ta) \in G, \ \forall\, t \in \mathbb{R}\}$. $\mathcal{g}$ is a vector subspace of $\mathrm{Lin}(V)$ with the property that if $a,b \in \mathcal{g}$ then $[a,b] \equiv ab - ba \in \mathcal{g}$. $\mathrm{Exp}(\mathcal{g})$ generates the largest connected Lie subgroup of G (Sagle and Walde [89] p. 141). The connected Lie subgroups of G are in one to one correspondence with the Lie subalgebras of $\mathcal{g}$, each group generated by the exponential of its corresponding algebra. The dimension of the vector space $\mathcal{g}$ equals the dimension of the manifold G. The elements of $\mathcal{g}$ can be put into a natural one to one correspondence with the tangent space to G at the identity and also with the left invariant vector fields on G. Every connected Lie subgroup G of $GL(V)$ (i.e. G is the intersection of all Lie subgroups containing $\exp \mathcal{g}$) is orientation preserving, i.e. $\det g > 0$, $\forall\, g \in G$. If G is a volume preserving Lie subgroup, then every $a \in \mathcal{g}$ has $\mathrm{tr}(a) = 0$. If $g_o \in GL(V)$ then $G_o = g_o G g_o^{-1}$ is also a Lie group conjugate to G with conjugate Lie algebra $g_o \mathcal{g} g_o^{-1} = \mathcal{g}_o$. It is the Lie algebra of the group structure $g_o G(M)$. If $\mathcal{g}_x$ is the Lie algebra of G_x we

have the naturally defined Lie algebra isomorphism $\tilde{r}_{\alpha x}: \mathcal{g}_x \to \mathcal{g}$, $x \in U_\alpha$ by $\tilde{r}_{\alpha x}(a) = r_{\alpha x} \circ a \circ r_{\alpha x}^{-1}$.

(II.2) Riemann and Lorentz Structures

There are particular group structures on a manifold which are of special interest. If $G(M) = (M, G, U, V, G)$ is a group structure and if there exists $I \in \mathrm{Sym}^+(V, V^*)$ [a linear isomorphism from V to V^* naturally identified with a positive definite symmetric bilinear form on V] such that $g^* \circ I \circ g = I$ for all $g \in G$, we say that the group structure is *Riemannian*. I determines a unique smooth invariant covariant tensor field of order 2 on M called the *metric tensor*.

If $\mathrm{Invlin}(V, V^*)$ denotes the set of all linear isomorphisms from V to V^* as in Noll [77] then an element $I \in \mathrm{Sym}(V, V^*) \cap \mathrm{Invlin}(V, V^*)$ is said to be a *Lorentz inner product* if the following condition holds. There exists a basis $v = \{v_1, v_2, \cdots, v_{n+1}\}$ of V (assumed to be of dimension $n + 1$) such that $I(v_i) = v_i^*$, $1 \le i \le n$ and $I(v_{n+1}) = -v_{n+1}^*$. Such a basis v is called a *Lorentz basis*. If $g: V \to V$ is a linear isomorphism which maps (componentwise) a Lorentz basis to a Lorentz basis, we say g is a Lorentz map. The set of all Lorentz maps is called the *Lorentz group*. We let L_I denote the group of all Lorentz maps corresponding to I. It can be shown that L_I is a Lie subgroup of GL(V).

(II.3) Abstract Minkowski and Newtonian Spaces

An *Abstract Minkowski space* is a vector space V equipped with a Lorentz inner product I, written (V,I). If $v \in V$ we say that (relative to I) v is *space-like* if $\langle Iv, v\rangle > 0$, *time-like* if $\langle Iv, v\rangle < 0$ and *null* if $\langle Iv, v\rangle = 0$. Putting $\eta_{ij} = 0$, $i \neq j$,

$\eta_{ii} = 1$, $i = 1,\cdots,n$, $\eta_{n+1,n+1} = -1$, i.e. $(\eta_{ij}) = \mathrm{diag}(1,1,\cdots,1,-1)$, then $\langle Iv_i,v_j\rangle = \eta_{ij}$ for any Lorentz basis, so v_{n+1} is time-like and v_i, $1 \le i \le n$ are space-like.

If $v_1,v_2 \in V$ are arbitrary, and $\ell: V \to V$ is a Lorentz isomorphism, it is easy to see that $\langle Iv_1,v_2\rangle = \langle I\ell v_1,\ell v_2\rangle$. Hence $\ell^* I\ell = I$. We can identify V with V^* using I and will do this from now on without further specifying it. A subspace of V is said to be *space-like* if every vector in it is space-like except for zero. A subspace is *time-like* if it contains a time-like vector and *null* if it is neither space-like or time-like.

One always has $\det \ell = \pm 1$ for $\ell \in L_I$ since $\ell^{-1} = \ell_*$ when we use I to identify V^* and V. ℓ is said to be *proper* if $\det \ell = +1$ and *orthochronous* if $\langle Iv,\ell v\rangle < 0$ for all time-like $v \in V$. The proper Lorentz group L_{I+} consists of all proper Lorentz isomorphisms. We shall prove later that ℓ is orthochronous if and only if $\langle Iv,\ell v\rangle < 0$ for any one time-like $v \in V$.

Remarks: 1) *A subspace of* V *generated by space-like vectors is space-like if the vectors are orthonormal.*

2) *There do not exist two orthogonal time-like vectors, i.e.,* $\langle Iv,v\rangle < 0$, $\langle Iw,w\rangle < 0 \Rightarrow \langle Iw,v\rangle \ne 0$. In fact if w is orthogonal to a time-like vector v, then w is space-like or zero.

Lemma: *Let* v, w, x *be three time-like vectors in* V. *Suppose* $\langle Iv,w\rangle < 0$ *and* $\langle Iw,x\rangle < 0$. *Then* $\langle Iv,x\rangle < 0$.

Proof: Let $w_1,w_2,\cdots,w_{n+1}$ be a Lorentz basis for V with $w_{n+1} = w$. (We assume $\langle Iw,w\rangle = -1$ without loss of generality). Then $Iw_i = \eta_i w_i^*$ where $\eta_1 = \eta_2 = \cdots = \eta_n = 1$, $\eta_{n+1} = -1$. Let $v = \sum_{i=1}^{n+1} \alpha_i w_i$ and

$x = \sum_{i=1}^{n+1} \beta_i w_i$. Then we have v time-like implies $\alpha_{n+1}^2 > \sum_{i=1}^{n} \alpha_i^2$ and x time-like implies $\beta_{n+1}^2 > \sum_{i=1}^{n} \beta_i^2$. Also $\langle Iv,w\rangle = \langle Iv,w_{n+1}\rangle < 0$ implies $\alpha_{n+1} > 0$ and $\langle Iw,x\rangle < 0$ implies $\beta_{n+1} > 0$. Therefore $\langle Iv,x\rangle = \sum_{i=1}^{n} \alpha_i\beta_i - \alpha_{n+1}\beta_{n+1} \leq \sqrt{\sum_{i=1}^{n} \alpha_i^2}\sqrt{\sum_{i=1}^{n} \beta_i^2} - \alpha_{n+1}\beta_{n+1} < \alpha_{n+1}\beta_{n+1} - \alpha_{n+1}\beta_{n+1} = 0$. From this result we quickly obtain,

Proposition 1. (a) *If there exist time-like vectors* v *and* w *with* $\langle Iv,w\rangle < 0$ *and* $\langle Iv,\ell w\rangle < 0$ *where* $\ell \in L_I$ *then* ℓ *is orthochronous.*

(b) *If* $\ell \in L_I$ *is orthochronous, then for any time-like vectors* v', w' *with* $\langle Iv',w'\rangle < 0$ *we have* $\langle Iv',\ell w'\rangle < 0$.

Proof: Suppose v and w exist as specified in (a) and let v', w' be time-like and arbitrary with $\langle Iv',w'\rangle < 0$. Since time-like vectors are never orthogonal (Remark 2) we have $\langle Iv,v'\rangle > 0$ or $\langle Iv,v'\rangle < 0$. If $\langle Iv,v'\rangle > 0$ we can replace v by $-v$ and w by $-w$ and (a) still holds, so we may take $\langle Iv,v'\rangle < 0$ without loss of generality. By the lemma, $\langle Iw,\ell w\rangle < 0$, $\langle Iv',w\rangle < 0$ and $\langle Iv',\ell w\rangle < 0$. Since $\langle Iv',w'\rangle < 0$ we see that $\langle Iw',\ell w\rangle < 0$ and $\langle Iw',w\rangle < 0$. But $\langle Iw',w\rangle = \langle I\ell w',\ell w\rangle$ so $\langle I\ell w',w'\rangle < 0$ by the lemma, and $\langle I\ell w',w\rangle < 0$ so $\langle I\ell w',v'\rangle < 0$. Hence $\langle Iv',\ell w'\rangle < 0$ as required. We define $L_I^{\uparrow}$ to be the set of all orthochronous Lorentz automorphisms of V and $L_{I+}^{\uparrow} = L_{I+} \cap L_I^{\uparrow}$ to be the proper orthochronous ones. Then,

Proposition 2: $L_I^{\uparrow}$ *is a group called the* orthochronous Lorentz group, *and therefore* $L_{I+}^{\uparrow}$ *is a group called the* restricted or special Lorentz group.

Proof: Let ℓ_1 and ℓ_2 be orthochronous, and consider $\ell_1 \circ \ell_2$. For

time-like vectors $v, w \in V$ with $\langle Iv, w\rangle < 0$ we have $\langle Iv, \ell_1 \circ \ell_2 w\rangle =$ $\langle Iv, \ell_1 x\rangle$ where $x = \ell_2 w$. Now $\langle Iv, \ell_2 w\rangle < 0$ since $\langle Iv, w\rangle < 0$ and ℓ_2 is orthochronous, i.e. $\langle Iv, x\rangle < 0$ so then $\langle Iv, \ell_1 x\rangle < 0$ since ℓ_1 is orthochronous. Hence so is $\ell_1 \circ \ell_2$. Furthermore, if $\ell \in L_I^{\uparrow}$ then $\langle Iv, \ell w\rangle < 0$ so $\langle Iw, \ell^* v\rangle = \langle Iw, \ell^{-1} v\rangle < 0$ so $\ell^{-1} \in L_I^{\uparrow}$ so $L_I^{\uparrow}$ is indeed a group.

Remarks: 3) *If a subspace of* V *contains a space-like and a time-like vector, it must contain a non-zero null vector.*

4) *If a time-like and null vector are orthogonal then the null vector is zero.*

5) *If a non-zero subspace of* V *has all non-zero vectors time-like, then it is one dimensional.*

6) *Any null subspace of* V *is one dimensional and generated by a single null vector.*

If (V, I) is an abstract Minkowski space and G is either L_I, $L_I^{\uparrow}$, L_{I+} or $L_{I+}^{\uparrow}$, all of which are Lie subgroups of $GL(V)$ as can be verified, then $g^* I g = I$ for all $g \in G$. In fact, we may take G to be any Lie subgroup of L_I. Then if $\mathcal{G}(M) = (M, \mathcal{G}, \mathcal{U}, V, G)$ is a group structure on a manifold M we say the group structure is *Lorentzian,* and the Lorentz inner product I on V can be lifted uniquely to an invariant smooth covariant second order symmetric tensor field on M called the (Lorentz) *metric tensor.* If g_{ab} is the component form of this tensor, we have that $\left(M_x, (g_{ab})\big|_x\right)$ is an abstract Minkowski space at x isomorphic to (V, I) for each x.

It is also worth considering the Abstract Newtonian space at this time. Let V be an $n+1$ dimensional vector space equipped with a

non-zero element $u^* \in V^*$, and hence a distinguished subspace $V^\perp$ of dimension n with $\langle u^*, v\rangle = 0$, $\forall\, v \in V^\perp$. Suppose also there is a distinguished element $\gamma \in \mathrm{Sym}(V^*, V)$ with $\gamma(u^*) = 0$ and $\langle\gamma(v^*), v^*\rangle > 0$ if $v^* \neq \lambda u^*$ for any real λ. We say that the triple (V, u^*, γ) is an *Abstract Newtonian space.* Clearly $\ker(\gamma)$ is the space generated by u^* and $\mathrm{Im}(\gamma) = V^\perp$. A basis $v = (v_1, v_2, \cdots, v_{n+1})$ for V is said to be a *Galilei basis* if $v^*_{n+1} = -u^*$ and $\gamma = v_1 \otimes v_1 + \cdots + v_n \otimes v_n$ where $(v^*_1, v^*_2, \cdots, v^*_{n+1}) = v^*$ is the dual basis to v and we define $a \otimes b \in \mathrm{Lin}(V^*, V)$ by $(a \otimes b)(c^*) = \langle b, c^*\rangle a$, $a, b \in V$. A Galilei automorphism $g\colon V \to V$ is a linear map that transforms one Galilei basis componentwise into another. The Galilei group G_V is the group of all Galilei isomorphisms from V to V, which is also a Lie group, and a subgroup of $\mathrm{Unim}(V)$. We write $G_{V+} = G_V \cap SL(V)$ and call this the *proper Galilei group.* The restricted (or special) Lorentz group and the proper Galilei group are connected Lie groups.

If (V, u^*, γ) is an abstract Newtonian space and $G(M) = (M, G, U, V, G)$ is a group structure on a manifold M where G is a Lie subgroup of G_V, then u^* can be lifted to a smooth invariant 1-form on M (which we denote by u_a in component form) and γ can be lifted to a smooth invariant symmetric contravariant second order tensor field on M (which we denote by γ^{ab}). We say M is equipped with a *Galilei structure,* and write (M, u_a, γ^{ab}) where $u_a\gamma^{ab} = 0$.

If we take $G = G_V$ or G_{V+} then we can explicitly evaluate the Lie algebra g. Since g consists of maps from V to V we can take a Galilei basis and find the elements of g are represented by matrices of the form shown below. Likewise for a Lorentz structure on M, taking $G = L_I$, L_{I+}, $L^\uparrow_I$ or $L^\uparrow_{I+}$ and using a Lorentz basis,

the corresponding Lie algebra $\mathcal{g}$ can be described by the set of all matrices as shown below.

$$\text{Galilei case,}\quad \begin{pmatrix} 0 & a_3 & -a_2 & b_1 \\ -a_3 & 0 & a_1 & b_2 \\ a_2 & -a_1 & 0 & b_3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \text{Lorentz case,}\quad \begin{pmatrix} 0 & a_3 & -a_2 & b_1 \\ -a_3 & 0 & a_1 & b_2 \\ a_2 & -a_1 & 0 & b_3 \\ b_1 & b_2 & b_3 & 0 \end{pmatrix} \qquad \text{(II.3.1)}$$

Here we have taken $n + 1 = 4$, and since there are 6 parameters, the dimension of the vector space $\mathcal{g}$ is 6 in each case. If $a,b \in \mathcal{g}$ are represented by matrices A and B respectively, then $[a,b]$ is represented by $AB - BA$ which is a matrix of the same form as the above in each case. We can, in this way, naturally associate $\mathcal{g}$ with the tangent space G_e to the Lie group G at the identity e and hence also with the left invariant vector fields on G. For further information on Galilei structures expecially, see Künzle [46].

Let $\mathcal{G}(M) = (M,\mathcal{G},\mathcal{U},V,G)$ be a group structure on M. We say that $\mathcal{G}'(M) = (M,\mathcal{G}',\mathcal{U}',V,G')$ is a *subgroup structure* of $\mathcal{G}(M)$ if G'_x is a Lie subgroup of G_x $\forall\, x \in M$ $(\mathcal{G} = \{G_x,\ x \in M\},\ \mathcal{G}' = \{G'_x,\ x \in M\})$ and if G' is a Lie subgroup of G and if $\mathcal{U} = \{(U_\alpha,r_\alpha),\ \alpha \in \mathcal{A}\}$ then $\mathcal{U}' = \{(U_\alpha,r_\alpha),\ \alpha \in \mathcal{A}'\}$ where $\mathcal{A}' \subset \mathcal{A}$. Any subgroup structure of a Riemannian [Lorentzian, Galileian] group structure is again a Riemannian [Lorentzian, Galileian] group structure. In general, the subgroup structure will possess a wider possible variety of different elements I or (u^*,γ) satisfying the group structure conditions than the original structure. For the Galilei structure it is worth noting that these conditions for $g \in G_V$ are $g \circ \gamma \circ g^* = \gamma$ and $g^*(u^*) = u^*$ where $g^*: V^* \to V^*$, $g: V \to V$.

The concept of a group structure can be applied immediately to

the theory of a smooth materially uniform simple body (Wang [108] p. 46) and permits us to describe the properties of a body manifold in terms of a vector space. The material connections on the body and space time connections can be described in terms of the concept of a connection on the group structure, which we now investigate in some detail.

(II.4) Connections on a Group Structure

In an abstract and general sense, a connection can be looked upon as a smooth field of horizontal subspaces following Kobayashi and Nomizu [44] and Wang [108].

(II.5) The Fundamental Fields on the Principal Bundle

Let $\mathcal{G}(M)$ be a group structure on the manifold M and let $E(M,\mathcal{U})$ be the principal fibre space associated with $\mathcal{G}(M)$ as we have defined it. Then $E(\{x\},\mathcal{U}) = L_x(\mathcal{U}_0)$ is the fibre in the principal bundle associated with $x \in M$. A natural differentiable structure appears on $L_x(\mathcal{U}_0)$ treated as a manifold making each $\psi_{\alpha x}: G \to L_x(\mathcal{U}_0)$ a diffeomorphism $\forall\alpha$ such that $x \in U_\alpha$. An element $a \in \mathfrak{g}$, the Lie algebra of the structure group G can also be viewed as a left invariant vector field on G. Then for each $a \in \mathfrak{g}$ we can associate with a, a vector field $\bar{a}$ on $L_x(\mathcal{U}_0)$ for each $x \in M$.

We call $\bar{a}$ the *fundamental field* associated with $a \in \mathfrak{g}$. It is defined in the natural way from the diagram below as $\bar{a} = \psi_{\alpha x*} \circ a \circ \psi_{\alpha x}^{-1}$.

$$\begin{tikzcd}
TG \arrow[r, "\psi_{\alpha x*}"] \arrow[dr, phantom, "\text{(Left invariant vector field)}"] & TL_x(\mathcal{U}_0) \\
G \arrow[r, "\psi_{\alpha x}"] \arrow[u, "a"] & L_x(\mathcal{U}_0) \arrow[u, "\bar{a}"']
\end{tikzcd}$$

We only have to prove that $\bar{a}$ as defined here is independent of the choice of α, such that $x \in U_\alpha$. Choose also β with $x \in U_\beta$. We can use the left

invariant property of $a \in g$ to prove this. Left invariance means that for each $g \in G$, $L_{g*} \circ a = a \circ L_g$ as illustrated in the diagram. Taking

$$\begin{tikzcd} TG \arrow[r, "L_{g*}"] & TG \\ G \arrow[u, "a"] \arrow[r, "L_g"] & G \arrow[u, "a"'] \end{tikzcd}$$

$L_g = \overline{G}_{\alpha\beta}(x) = \psi_{\alpha x}^{-1} \circ \psi_{\beta x}$ we have $\psi_{\alpha x*}^{-1} \circ \psi_{\beta x*} \circ a = a \circ \psi_{\alpha x}^{-1} \circ \psi_{\beta x}$ or, since all the ψ's and derivatives are invertible, $\overline{a} = \psi_{\alpha x*} \circ a \circ \psi_{\alpha x}^{-1} = \psi_{\beta x*} \circ a \circ \psi_{\beta x}^{-1}$. Consequently, the fundamental field is well defined, independent of the choice of α.

We can lift $\overline{a}$ in a natural way to a cross section $\tilde{a}$ on the principal bundle. We let i_x denote the inclusion map from $L_x(\mathcal{U}_0)$ to

$$\begin{tikzcd} TL_x(\mathcal{U}_0) \arrow[r, "i_{x*}"] & TE(M,\mathcal{U}) \\ L_x(\mathcal{U}_0) \arrow[u, "\overline{a}"] \arrow[r, "i_x"] & E(M,\mathcal{U}) \arrow[u, "\tilde{a}"'] \end{tikzcd}$$

$E(M,\mathcal{U})$. Since $E(M,\mathcal{U})$ is a disjoint union of the $L_x(\mathcal{U}_0)$ for $x \in M$ we define $\tilde{a}(\phi_{\alpha x}) = i_{x*} \circ \overline{a}(\phi_{\alpha x})$ for all $\phi_{\alpha x} \in L_x(\mathcal{U}_0)$ and all $x \in M$, (α is such that $x \in U_\alpha$). We let $\tilde{g} = \{\tilde{a} \mid a \in g\}$ and define $[\tilde{a},\tilde{b}] = \mathcal{L}_{\tilde{a}}\tilde{b}$. Then just as $[a,b] = \mathcal{L}_a b$ in g where a and b are interpreted as left invariant vector fields on G, the parallel relation for $\tilde{g}$ gives us $[\tilde{a},\tilde{b}] = \widetilde{[a,b]}$. Thus the Lie algebras g and $\tilde{g}$ are isomorphic Lie algebras.

Now we can look upon $TE(M,\mathcal{U}) = \displaystyle\bigcup_{\phi_{\alpha x} \in E(M,\mathcal{U})} E(M,\mathcal{U})_{\phi_{\alpha x}}$ as a disjoint union of the respective tangent spaces. We have that $\{\tilde{a}(\phi_{\alpha x}) \mid \tilde{a} \in \tilde{g}\} \subset$

$$\begin{tikzcd} T(M) & TE(M,\mathcal{U}) \arrow[l, "\pi_*"'] \arrow[d, "\rho"] \\ M \arrow[from=u, "P"'] & E(M,\mathcal{U}) \arrow[l, "\pi"'] \end{tikzcd}$$

$E(M,\mathcal{U})_{\phi_{\alpha x}}$ is a vector subspace of the same dimension as the Lie algebra, and it also equals $\operatorname{Ker} \pi_*$ at that point $\phi_{\alpha x}$. As a result $\pi_* \circ \tilde{a} = 0$ for all $\tilde{a} \in \tilde{g}$. A *horizontal subspace* of $E(M,\mathcal{U})_{\phi_{\alpha x}}$ is a vector subspace $H_{\phi_{\alpha x}}$ with the property that $E(M,\mathcal{U})_{\phi_{\alpha x}} =$

$H_{\phi_{\alpha x}} \oplus V_{\phi_{\alpha x}}$ where $V_{\phi_{\alpha x}} = \ker\left(\pi_* | E(M,\mathcal{U})_{\phi_{\alpha x}}\right) = \{\tilde{a}(\phi_{\alpha x}) | \tilde{a} \in \tilde{g}\}$. An *abstract global connection* on M is a smooth map $\gamma\colon V \times E(M,\mathcal{U}) \to TE(M,\mathcal{U})$ with $\pi_* \circ \gamma(v,\phi_{\alpha x}) = \phi_{\alpha x}(v)$, $\rho \circ \gamma(v,\phi_{\alpha x}) = \phi_{\alpha x}$ and $\gamma_{\phi_{\alpha x}}\colon V \to E(M,\mathcal{U})_{\phi_{\alpha x}}$ is a linear one to one map whose image $H_{\phi_{\alpha x}}$ is a horizontal subspace. Thus a connection determines a smooth field of horizontal subspaces, and also conversely. In the obvious way we define an *abstract local connection* $\gamma\colon V \times E(U,\mathcal{U}) \to TE(U,\mathcal{U})$ on neighborhoods in M.

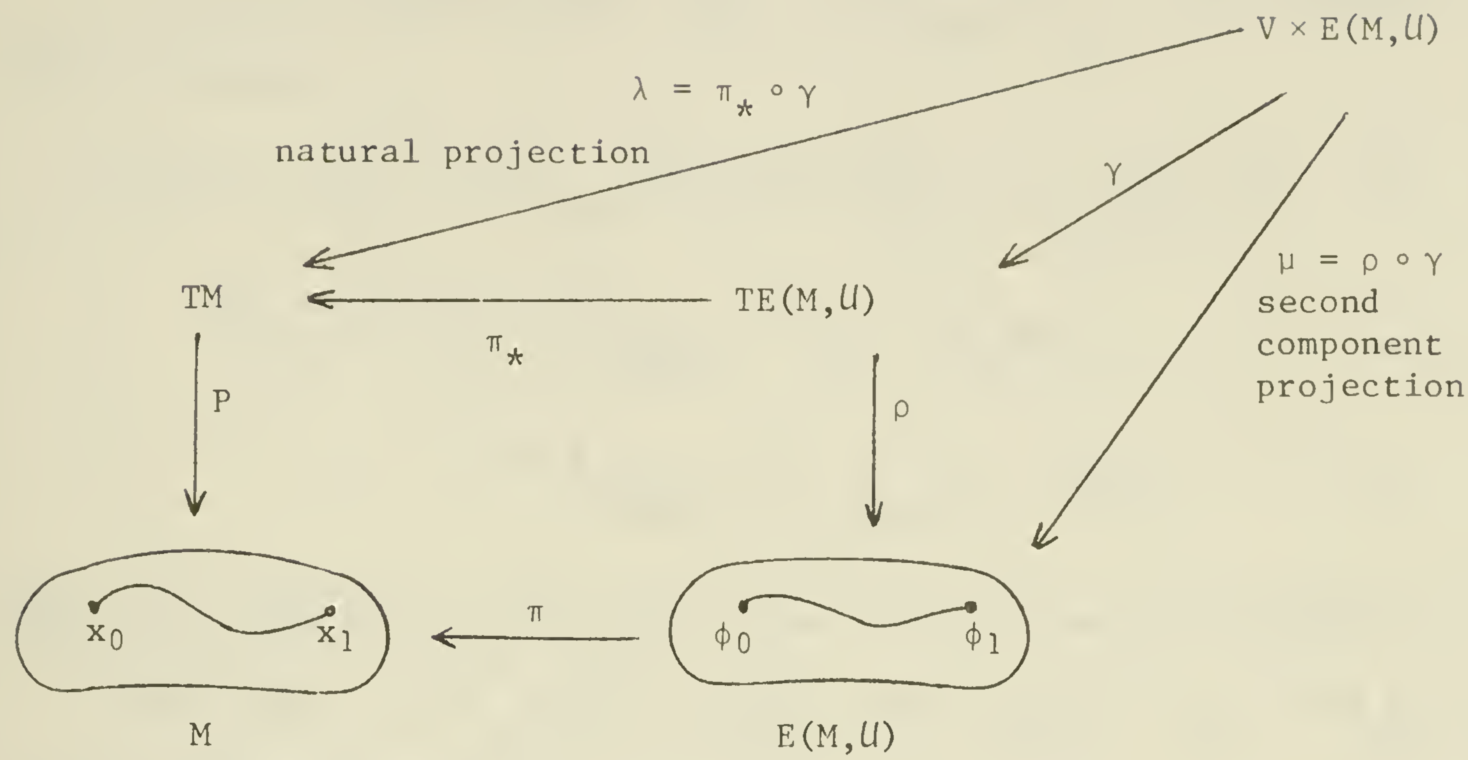
$$\begin{tikzcd}
V \times E(M,\mathcal{U}) \arrow[d, "\lambda"] \arrow[dr, "\gamma"] & \\
TM \arrow[d, "P"] & TE(M,\mathcal{U}) \arrow[l, "\pi_*"'] \arrow[d, "\rho"] \\
M & E(M,\mathcal{U}) \arrow[l, "\pi"']
\end{tikzcd}$$

We can define a natural projection map $\lambda\colon V \times E(M,\mathcal{U}) \to TM$ by $\lambda(v,\phi_{\alpha x}) = \phi_{\alpha x}(v)$. This makes the diagram on the left commutative in every case. If a smooth field $H_{\phi_{\alpha x}}$ is specified, we define $\gamma_{\phi_{\alpha x}}\colon V \to E(M,\mathcal{U})_{\phi_{\alpha x}}$ for each $\phi_{\alpha x}$ by $\gamma_{\phi_{\alpha x}} = (\pi_{*\phi_{\alpha x}} | H_{\phi_{\alpha x}})^{-1} \circ \phi_{\alpha x}$, and this determines our connection γ. Of course, $\dim V_{\phi_{\alpha x}} = m = \dim g$ and $\dim H_{\phi_{\alpha x}} = n = \dim M$. Thus the connection can also be represented by a field of projection maps $h_{\phi_{\alpha x}}\colon E(M,\mathcal{U})_{\phi_{\alpha x}} \to V_{\phi_{\alpha x}}$ for each point $\phi_{\alpha x}$, such that $\ker h_{\phi_{\alpha x}} = H_{\phi_{\alpha x}}$. We have the natural vector space isomorphism $|_{\phi_{\alpha x}}\colon \tilde{g} \to V_{\phi_{\alpha x}}$ (evaluation map) which can be used to define $\omega_{\phi_{\alpha x}} = |_{\phi_{\alpha x}}^{-1} \circ h_{\phi_{\alpha x}}\colon E(M,\mathcal{U})_{\phi_{\alpha x}} \to \tilde{g}$, which is a linear onto map for each $\phi_{\alpha x}$. In addition we have the isomorphisms $\eta_{\phi_{\alpha x}} = (\pi_{*\phi_{\alpha x}} | H_{\phi_{\alpha x}})\colon H_{\phi_{\alpha x}} \to M_{\pi(\phi_{\alpha x})} = M_x$, $x \in U_\alpha$. Clearly, then, $\gamma_{\phi_{\alpha x}} = \eta_{\phi_{\alpha x}}^{-1} \circ \phi_{\alpha x}$.

(II.6) Horizontal Lift and Parallel Transport

Let $x\colon [0,1] \to M$ be a smooth curve in M with $x_t = x(t)$, $x_0 = x(0)$, $x_1 = x(1)$, and suppose γ is an abstract local connection defined on a neighborhood of the curve x. Let $\phi_0 \in E(M,\mathcal{U})$ be arbitrary with $\pi(\phi_0) = x_0$, i.e. $\phi_0 \in L_{x_0}(\mathcal{U}_0)$. [In case γ is not global, simply

replace M by U every time it occurs.]



Then there exist unique functions $v\colon [0,1] \to V$ and $\phi\colon [0,1] \to E(M,U)$ such that $\lambda(v_t,\phi_t) = \dot{x}_t = \left.\frac{dx}{dt}\right|_t$ and $\gamma(v_t,\phi_t) = \dot{\phi}_t$, $\forall\, t \in [0,1]$ where $\phi(0) = \phi_0$. The curve ϕ, clearly a solution to a differential equation with a smooth function, satisfies $\pi(\phi_t) = x_t$ and is called the *horizontal lift* of x with starting point ϕ_0. Of course, $v_t = \phi_t^{-1}(\dot{x}_t)$ and $\lambda_{\phi_t} = \phi_t\colon V \to M_{x_t}$, and $\pi_*(\dot{\phi}_t) = \dot{x}_t$.

If x is fixed the curve ϕ will depend on the choice of the initial point ϕ_0. We can therefore define a one parameter family of diffeomorphisms of the fibres, namely $\rho_t\colon L_{x_0}(U_0) \to L_{x_t}(U_0)$ for $t \in [0,1]$ by $\rho_t\big(\phi(0)\big) = \phi(t)$. They are called the *parallel transports* along x with respect to γ. If $x_t \in U_\alpha$ for $t \in [0,\varepsilon]$ for some $0 < \varepsilon \le 1$ we can define a class of transformations on the group G, namely $\psi^{-1}_{\alpha x_t} \circ \rho_t \circ \psi_{\alpha x_0}$: $G \to G$ which is a diffeomorphism, although not in general a left multiplication. We say that γ is a G-*connection* if for every curve x the transformation above on G is a left multiplication by an element of G,

so that $\psi^{-1}_{\alpha x_t} \circ \rho_t \circ \psi_{\alpha x_0} = L_{g_t}$.

If $x_0 \in U_\beta$ also we have $\psi^{-1}_{\beta x_t} \circ \rho_t \circ \psi_{\beta x_0} = L_{\bar{g}_t}$. The comparison is made by taking

$$\bar{G}^{-1}_{\alpha\beta}(x_t) \circ \psi^{-1}_{\alpha x_t} \circ \rho_t \circ \psi_{\alpha x_0} \circ \bar{G}_{\alpha\beta}(x_0) = \bar{G}^{-1}_{\alpha\beta}(x_t) \circ L_{g_t} \circ \bar{G}_{\alpha\beta}(x_0), \quad \text{or}$$

$$\psi^{-1}_{\beta x_t} \circ \rho_t \circ \psi_{\beta x_0} = L_{G^{-1}_{\alpha\beta}(x_t)} \circ L_{g_t} \circ L_{G_{\alpha\beta}(x_0)}$$

$$= L_{G^{-1}_{\alpha\beta}(x_t) \cdot g_t \cdot G_{\alpha\beta}(x_0)} = L_{\bar{g}_t} .$$

Hence $\bar{g}_t = G^{-1}_{\alpha\beta}(x_t) \cdot g_t \cdot G_{\alpha\beta}(x_0)$ where "$\cdot$" is group multiplication.

Recall that $\rho_t \colon L_{x_0}(\mathcal{U}_0) \to L_{x_t}(\mathcal{U}_0)$ and $L_x(\mathcal{U}_0) = \{\phi_{\alpha x} \colon V \to M_x \mid x \in U_\alpha, \alpha \in A\}$. Thus we can write

$$\underbrace{\rho_t(\phi_{\alpha x_0})}_{\phi_{\alpha x_0} \colon V \to M_{x_0}} \quad = \quad \underbrace{\phi_{\alpha_t x_t}}_{\phi_{\alpha_t x_t} \colon V \to M_{x_t}}$$

so ρ_t maps one isomorphism to another isomorphism. From this we obtain the tangent space parallel transport on M namely $\tilde{\rho}_t \colon M_{x_0} \to M_{x_t}$ by taking $\tilde{\rho}_t = \phi_{\alpha_t x_t} \circ \phi^{-1}_{\alpha x_0} = [\rho_t(\phi_{\alpha x_0})] \circ \phi^{-1}_{\alpha x_0}$. $\tilde{\rho}_t$ is an isomorphism that is well defined if and only if $[\rho_t(\phi_{\alpha x_0})] \circ \phi^{-1}_{\alpha x_0} = [\rho_t(\phi_{\beta x_0})] \circ \phi^{-1}_{\beta x_0}$ for each α, β with $x_0 \in U_\alpha \cap U_\beta$. Alternatively we can write $\rho_t(\phi_{\alpha x_0}) = \tilde{\rho}_t \circ \phi_{\alpha x_0} \colon V \to M_{x_t}$, and this characterizes the type of parallel transport ρ_t which allows $\tilde{\rho}_t$ to exist.

A *connection on the group structure* $G(M)$ is a connection γ whose tangent space parallel transports ρ_t exist along any curve, and the corresponding isomorphisms are always structure preserving.

Now look at the following diagram. We have $\psi_{\alpha x}(g) = \phi_{\alpha_g x}$ where $g \in G$ and $\phi_{\alpha_g x} \in L_x(\mathcal{U}_0)$. Take $v \in V$. Then $\phi_{\alpha x}(gv) = \phi_{\alpha_g x}(v) = [\psi_{\alpha x}(g)](v)$

$$\begin{tikzcd}
 & V \arrow[dl, "g"'] \arrow[dd, "\epsilon"', "\mathrm{id}_V"] & G \arrow[dd, "\psi_{\alpha x}"] \\
V \arrow[dd, "\phi_{\alpha x}"'] & & \\
 & V \arrow[dl, "\phi_{\alpha_g x} \in L_x(\mathcal{U}_0)"] & {} \\
M_x & &
\end{tikzcd}$$

which is the original definition of $\psi_{\alpha x}$ as it was given.

We clearly have $\phi_{\alpha x} \circ g = \phi_{\alpha_g x}$ and hence $\phi_{\alpha_{g_1} x} \circ g_2 = \phi_{\alpha_{g_1 g_2} x} = \phi_{\alpha x} \circ g_1 \circ g_2$. The particular choice of α_g, $g \in G$ fixed here may not be unique. We can easily see that $\psi_{\alpha_{g_1} x}(g_2) = \phi_{\alpha_{g_1 g_2} x}$ for any $g_1, g_2 \in G$. Using this we can prove that $[\psi_{\alpha x}^{-1} \circ \psi_{\beta x}](g) = (\phi_{\alpha x}^{-1} \circ \phi_{\beta x}) \cdot g$ where $\beta = \alpha_{g_1}$ for instance. Since $\psi_{\alpha x}(g_1 g) = \phi_{\alpha_{g_1 g} x}$ we see $(\psi_{\alpha x}^{-1} \circ \psi_{\beta x})(g) = \psi_{\alpha x}^{-1}\left(\psi_{\alpha_{g_1} x}(g)\right) = \psi_{\alpha x}^{-1}(\phi_{\alpha_{g_1 g} x}) = g_1 g$ and therefore $\psi_{\alpha x}^{-1} \circ \psi_{\beta x} = L_{g_1} = \bar{G}_{\alpha\beta}(x)$. But $\phi_{\alpha x}^{-1} \circ \phi_{\alpha_{g_1} x} = g_1 = \phi_{\alpha x}^{-1} \circ \phi_{\beta x} = G_{\alpha\beta}(x)$, so we indeed obtain the required equivalence. This simply gives a detailed proof of a fact we have known all along. The purpose for this digression is to introduce the required notation for the following analysis.

<u>Proposition</u>: γ *is a* G-*connection if and only if* γ *is a connection on the group structure.*

<u>Proof</u>: ($\Rightarrow$) We have $\rho_t = \psi_{\alpha x_t} \circ L_{g_t} \circ \psi_{\alpha x_0}^{-1}$ and have to show that $\rho_t(\phi_{\beta x_0}) = \tilde{\rho}_t \circ \phi_{\beta x_0}$ for some structure preserving isomorphism $\tilde{\rho}_t : M_{x_0} \to M_{x_t}$. Let $\beta = \alpha_g$, $g \in G$. Then

$$\rho_t(\phi_{\beta x_0}) = \rho_t(\phi_{\alpha_g x_0}) = \psi_{\alpha x_t} \circ L_{g_t} \circ \underbrace{\psi_{\alpha x_0}^{-1}(\phi_{\alpha_g x_0})}_{= g}$$

$$= \psi_{\alpha x_t}(g_t g).$$

If we let $\tilde{\rho}_t = [\psi_{\alpha x_t}(g_t g)] \circ \phi_{\beta x_0}^{-1} = [\psi_{\alpha x_t}(g_t g)] \circ \phi_{\alpha_g x_0}^{-1}$ $\quad (= [\rho_t(\phi_{\beta x_0})] \circ \phi_{\beta x_0}^{-1})$

then $\tilde{\rho}_t = [\psi_{\alpha x_t}(g_t g)] \circ g^{-1} \circ \phi_{\alpha x_0}^{-1} = [\psi_{\alpha x_t}(g_t)] \circ \phi_{\alpha x_0}^{-1}$ which is independent

of g and hence β. Thus $\tilde{\rho}_t$ is well defined. Since $\tilde{\rho}_t = \phi_{\alpha_{g_t} x_t} \circ \phi^{-1}_{\alpha x_0}$ it is therefore also structure preserving.

($\Leftarrow$) Since $\tilde{\rho}_t$ is structure preserving, we have an equation of the form $\tilde{\rho}_t = \phi_{\alpha_{g_t} x_t} \circ \phi^{-1}_{\alpha x_0}$ along the curve x. Also $\rho_t(\phi_{\beta x_0}) = \tilde{\rho}_t \circ \phi_{\beta x_0}$ for each β. Hence

$$[\psi^{-1}_{\alpha x_t} \circ \rho_t \circ \psi_{\alpha x_0}](g) = [\psi^{-1}_{\alpha x_t} \circ \rho_t](\phi_{\alpha_g x_0}) = \psi^{-1}_{\alpha x_t}\left(\rho_t(\phi_{\alpha_g x_0})\right)$$

$$= \psi^{-1}_{\alpha x_t}(\tilde{\rho}_t \circ \phi_{\alpha_g x_0}) = \psi^{-1}_{\alpha x_t}(\phi_{\alpha_{g_t g} x_t}) = g_t g.$$

Observe that

$$\tilde{\rho}_t = \phi_{\alpha_{g_t} x_t} \circ \phi^{-1}_{\alpha x_0} = \underbrace{\phi_{\alpha_{g_t} x_t} \circ g}_{\phi_{\alpha_{g_t g} x_t}} \circ \underbrace{g^{-1} \circ \phi^{-1}_{\alpha x_0}}_{\phi^{-1}_{\alpha_g x_0}}$$

$$= \phi_{\alpha_{g_t g} x_t} \circ \phi^{-1}_{\alpha_g x_0} \quad \text{in the expression above.}$$

This completes the proof of the proposition.

(II.7) <u>The α-Coordinate Representation of γ</u>

Choosing $\alpha \in A$ we can examine in more detail the behavior of the connection γ for a particular bundle chart (U_α, ϕ_α). If $x \in U_\alpha$ is a point of M, we can take

$(x, \hat{v}) \in TU_\alpha$ (using ϕ_α as identifying diffeomorphism), $\hat{v} \in V$.

$(x,g) \in E(U_\alpha, \mathcal{U})$ (using ψ_α as the identifying diffeomorphism), $g \in G$.

$[(x,g),(\hat{v},a)] \in TE(U_\alpha, \mathcal{U})$. (x,g) is the base point and $(\hat{v},a)$ is the tangent vector. $\hat{v} \in V$ identifies $(x,\hat{v}) \in TU_\alpha$ and $a \in \mathfrak{g}$ identifies an element of TG at g since there is a natural isomorphism (L^{-1}_{g*}) between G_g and G_e where $e \in G$ is the identity, and the latter can

$$
\begin{tikzcd}
U_\alpha \times V \arrow[d, "\text{diffeomorphism}", "\phi_\alpha"'] & & V \times E(U_\alpha, \mathcal{U}) \arrow[dll, "\lambda"'] \arrow[dl, "\gamma"'] \arrow[ddl, "\mu"] \\
TU_\alpha \arrow[d, "P"] & TE(U_\alpha, \mathcal{U}) \arrow[l, "\pi_*"'] \arrow[d, "\rho"] & \\
U_\alpha & E(U_\alpha, \mathcal{U}) \; \left(= \bigcup_{x \in U_\alpha} L_x(\mathcal{U}_0)\right) \arrow[l, "\pi"'] & \\
 & U_\alpha \times G \arrow[u, "\psi_\alpha", "\text{diffeomorphism}"'] &
\end{tikzcd}
$$

$v, \hat{v} \in V, \quad \hat{v} = gv$

$P(x,\hat{v}) = x$

$\pi(x,g) = x$

$\rho[(x,g),(\hat{v},a)] = (x,g)$

$\pi_*[(x,g),(\hat{v},a)] = (x,\hat{v})$

$\lambda(v,x,g) = (x,\hat{v})$

$\mu(v,x,g) = (x,g)$

$\gamma(v,x,g) = [(x,g),(\hat{v},a)]$

be identified with the Lie algebra. Clearly also (v,x,g) is naturally [with α specified] identified with an element of $V \times E(U_\alpha, \mathcal{U})$. Thus our connection γ can be looked upon, in local coordinates as a map $\gamma_{\alpha x}: V \times G \to \mathfrak{g}$ for $x \in U_\alpha$. It is specifying the functional dependence of $a \in \mathfrak{g}$ which determines the nature of the abstract connection.

(II.8) <u>Change of Coordinate Relations</u>

For $x \in U_\alpha \cap U_\beta$ we can write $\gamma_{\alpha x}(v_\alpha, g_\alpha) = a_\alpha$ and $\gamma_{\beta x}(v_\beta, g_\beta) = a_\beta$. We clearly have $v_\alpha = v_\beta$, $\psi_{\alpha x}(g_\alpha) = \psi_{\beta x}(g_\beta)$, $\phi_{\alpha x}(\hat{v}_\alpha) = \phi_{\beta x}(\hat{v}_\beta)$ where $\hat{v}_\alpha = g_\alpha v_\alpha$ and $\hat{v}_\beta = g_\beta v_\beta$. Therefore $g_\beta = \overline{G}_{\beta\alpha}(x)(g_\alpha) = G_{\beta\alpha}(x) \cdot g_\alpha$ and $\hat{v}_\beta = G_{\beta\alpha}(x)(\hat{v}_\alpha)$, as defined in (II.1).

Now let us consider how a_α transforms to a_β. First of all, we need to consider the Lie algebra and group in a different light. G, first of all, is a subgroup and submanifold of $GL(V)$ and $\mathfrak{g}$ is a linear subspace of $Lin(V) = \{a: V \to V \mid a \text{ is linear}\}$. $\mathfrak{g}$ can be looked upon as the tangent space to G at the identity e, $(e:V \to V$ is the

identity map). The tangent space to G at g is obtained by left translating $\mathcal{g}$ by g, namely

$$G_g = g\mathcal{g} \ . \qquad \text{(II.8.1)}$$

$$\begin{tikzcd}[column sep=large]
{[(g_1,a_1),\ (g_2,a_2)]} \arrow[r] & (g_3 a_3) \\
(G\times \mathcal{g})\times(G\times \mathcal{g}) \arrow[r, "\mathrm{mult}_*"] \arrow[d] & G\times \mathcal{g} \arrow[d] \\
G\times G \arrow[r, "\mathrm{mult}"] & G
\end{tikzcd}$$

Now $g_3 = g_1 g_2$ and $\mathrm{mult}(g_1,g_2) = g_1 g_2$. Recall that $g_1 g_2$ is actually a composition $g_1 \circ g_2$ of linear invertible maps from V onto V.

Now, $\mathrm{mult}(g_1 + \eta g_1 a_1,\ g_2 + \eta g_2 a_2) = g_1 g_2 + \eta g_1 a_1 g_2 + \eta g_1 g_2 a_2$, (to first order in η)

so $\mathrm{mult}(g_1 + \eta g_1 a_1, g_2 + \eta g_2 a_2) - \mathrm{mult}(g_1,g_2) = \eta(g_1 a_1 g_2 + g_1 g_2 a_2)$

$$= \eta g_1 g_2 a_3$$

which implies

$$a_3 = g_2^{-1} a_1 g_2 + a_2 . \qquad \text{(II.8.2)}$$

This then defines the derivative of the multiplication operation in terms of the Lie algebra. By setting $a_1 = 0$ we see that the derivative of left translation leaves the Lie algebra element unchanged — an obvious result that we naturally expect, i.e.

$$\begin{tikzcd}
G\times \mathcal{g} \arrow[r, "L_{g_1 *}"] \arrow[d] & G\times \mathcal{g} \arrow[d] \\
G \arrow[r, "L_{g_1}"] & G
\end{tikzcd}
\qquad L_{g_1 *}(g_2,a_2) = (g_1 g_2, a_2) = (g_3,a_3).$$

On the other hand, setting $a_2 = 0$ gives us the operation of right translation by g_2.

$$\begin{tikzcd}
G \times \mathfrak{g} \arrow[r, "R_{g_2 *}"] \arrow[d] & G \times \mathfrak{g} \arrow[d] \\
G \arrow[r, "R_{g_2}"] & G
\end{tikzcd}$$

$$R_{g_2 *}(g_1,a_1) = (g_1g_2, g_2^{-1}a_1g_2) = (g_3,a_3).$$

We also see from this that if $a \in \mathfrak{g}$, $g \in G$, then $g^{-1}ag \in \mathfrak{g}$. We define this to be the *Adjoint map*, namely $ad_g : \mathfrak{g} \to \mathfrak{g}$, $g \in G$ by $ad_g(a) = gag^{-1}$.

Hence $R_{g_0^{-1}*}(g,a) = R^{-1}_{g_0 *}(g,a) = \left(gg_0^{-1},\ ad_{g_0}(a)\right)$ and $(L_{g_0} \circ R_{g_0^{-1}})_*(g,a) = \left(g_0gg_0^{-1}, ad_{g_0}(a)\right)$. We can define $Ad_{g_0} : G \to G$ as $Ad_{g_0}(g) = g_0gg_0^{-1}$ so $Ad_{g_0} = L_{g_0} \circ R_{g_0^{-1}}$. Then $Ad_{g_0 *} = ad_{g_0}$, i.e. $Ad_{g_0 *}(g,a) = \left(Ad_{g_0}(g), ad_{g_0}(a)\right)$.

$$\begin{tikzcd}
G \times \mathfrak{g} \arrow[r, "inv_*"] \arrow[d] & G \times \mathfrak{g} \arrow[d] \\
G \arrow[r, "inv"] & G
\end{tikzcd}$$

Similarly, if we let $(g_2,a_2) = inv_*(g_1,a_1)$ and use $mult_*\left((g_1,a_1),\ inv_*(g_1,a_1)\right) = (e,0)$ we can show that $inv_*(g,a) = (g^{-1}, -gag^{-1}) = \left(g^{-1}, -ad_g(a)\right)$. Notice that $ad_{g_1} \circ ad_{g_2} = ad_{g_1g_2}$. Now let us consider again the transformation of a_α into a_β.

$$\begin{tikzcd}
(x,\hat{v}_\beta,g_\beta,a_\beta) & & (x,\hat{v}_\alpha,g_\alpha,a_\alpha) \arrow[ll, "\overline{G}_{\beta\alpha *}"'] \\
T(U_\alpha \cap U_\beta) \times G \times \mathfrak{g} \arrow[r, "\psi_{\beta *}"'] \arrow[d, "{(P,\mu_1)}"'] & TE(U_\alpha \cap U_\beta, \mathcal{U}) \arrow[d, "\rho"] & T(U_\alpha \cap U_\beta) \times G \times \mathfrak{g} \arrow[l, "\psi_{\alpha *}"] \arrow[d] \\
(U_\alpha \cap U_\beta) \times G \arrow[r, "\psi_\beta"] & E(U_\alpha \cap U_\beta, \mathcal{U}) & (U_\alpha \cap U_\beta) \times G \arrow[l, "\psi_\alpha"'] \\
(x,g_\beta) & & (x,g_\alpha) \arrow[ll, "\overline{G}_{\beta\alpha} = \psi_\beta^{-1} \circ \psi_\alpha"]
\end{tikzcd}$$

$$\begin{tikzcd}
(U_\alpha \cap U_\beta) \times V \arrow[d, "\phi_\alpha"'] \arrow[dr, "G_{\beta\alpha *} \circ \phi_\alpha"] & & \\
T(U_\alpha \cap U_\beta) \arrow[r, "G_{\beta\alpha *}"'] \arrow[d, "P"'] & G \times \mathfrak{g} \arrow[d, "\mu_1", "\text{first comp. proj.}"'] \arrow[dr, "\mu_2", "\text{second comp. proj.}"'] & \\
U_\alpha \cap U_\beta \arrow[r, "G_{\beta\alpha}"] & G & \mathfrak{g}
\end{tikzcd}$$

$$\begin{tikzcd}
T(U_\alpha \cap U_\beta) \times G \times \mathfrak{g} \arrow[r, "\overline{G}_{\beta\alpha *}"] \arrow[d, "{(P,\mu_1)}"] & G \times \mathfrak{g} \arrow[d, "\mu_1"] \\
(U_\alpha \cap U_\beta) \times G \arrow[r, "\overline{G}_{\beta\alpha}"] & G
\end{tikzcd}$$

Now we have, $\overline{G}_{\beta\alpha}(x,g) = G_{\beta\alpha}(x)\circ g = \mathrm{mult}\big(G_{\beta\alpha}(x),g\big)$, and so

$$\overline{G}_{\beta\alpha*}\big(\underbrace{(x,\hat{v}_\alpha)}_{\text{representation}},(g,a)\big) = \mathrm{mult}_*\big(G_{\beta\alpha*}\circ\phi_\alpha(x,\hat{v}_\alpha),(g,a)\big), \quad \text{or}$$

$$\overline{G}_{\beta\alpha*}(x,\hat{v}_\alpha,g_\alpha,a_\alpha) = \mathrm{mult}_*\big(G_{\beta\alpha*}\circ\phi_\alpha(x,\hat{v}_\alpha),(g_\alpha,a_\alpha)\big).$$

In our formula (II.8.2) for mult_*, $a_1 = \mu_2\circ G_{\beta\alpha*}\circ\phi_\alpha(x,\hat{v}_\alpha)$, $g_1 = G_{\beta\alpha}(x)$, $g_2 = g_\alpha, a_2 = a_\alpha$, $a_3 = a_\beta$, $g_3 = g_1g_2 = G_{\beta\alpha}(x)g_\alpha$. Hence $a_\beta = a_3 = g_2^{-1}a_1g_2 + a_2 = g_\alpha^{-1}[\mu_2\circ G_{\beta\alpha*}\circ\phi_\alpha(x,\hat{v}_\alpha)]g_\alpha + a_\alpha$. If we let $\tilde{G}_{\beta\alpha*x}\colon V\to \mathcal{g}$ denote the linear map naturally defined from $\mu_2\circ G_{\beta\alpha*}\circ\phi_\alpha$ at x we can write

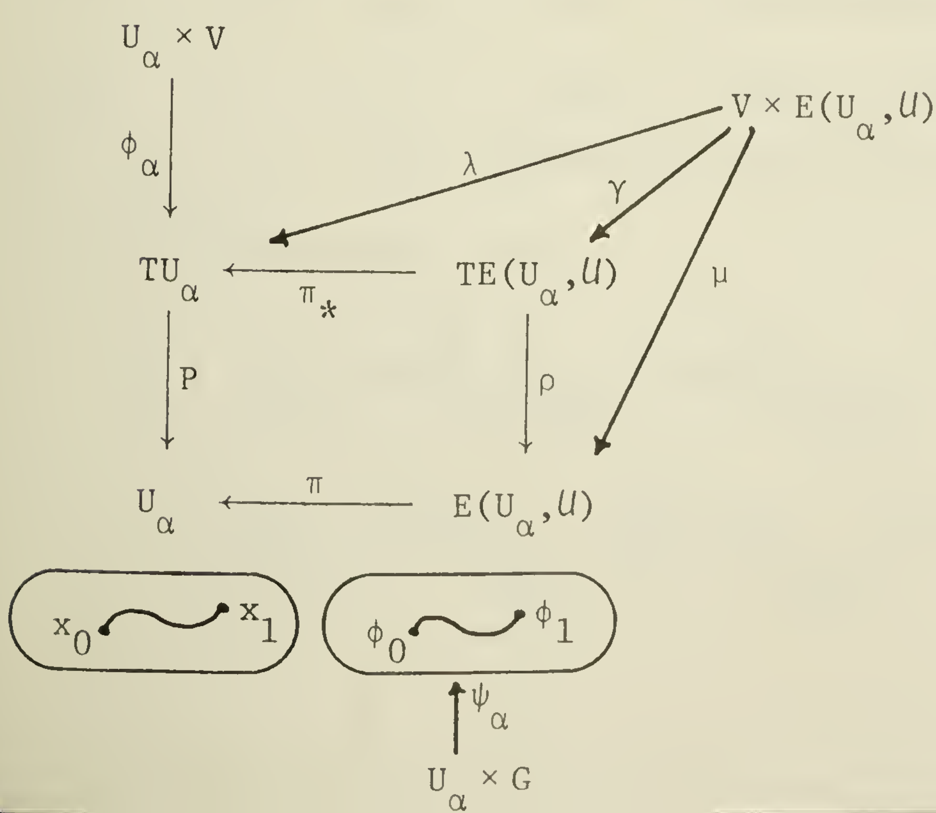
$$a_\beta = g_\alpha^{-1}\,\tilde{G}_{\beta\alpha*x}(\hat{v}_\alpha)g_\alpha + a_\alpha\ . \tag{II.8.3}$$

Of course $G_{\alpha\beta*} = \mathrm{inv}_*\circ G_{\beta\alpha*}$ and this is consistent when applied to the above relation as it interchanges the roles of α and β. We also see that

$$\tilde{G}_{\beta\alpha*x}(\hat{v}_\alpha) + G_{\alpha\beta}(x)\tilde{G}_{\alpha\beta*x}(\hat{v}_\beta)G_{\beta\alpha}(x) = 0. \tag{II.8.4}$$

(II.9) Parallel Transports in Coordinates

Consider a smooth curve $x\colon [0,1]\to U_\alpha$, $x(t) = x_t$.



Let $\dot{x}_t = \left.\frac{dx}{dt}\right|_t \in M_{x_t} \subset TU_\alpha$ and let $(x_t, \hat{v}_t) = \phi_\alpha^{-1}(\dot{x}_t)$ define $\hat{v}_t \in V$.

We use the α-coordinate representation here to express the operation of the various functions under consideration. If ϕ is the horizontal lift of x (with value ϕ_t at t) and initial point ϕ_0 we may let $(x_t, g_t) = \psi_\alpha^{-1}(\phi_t)$ define g_t. If $v_t = g_t^{-1}\hat{v}_t$ clearly $\lambda(v_t, x_t, g_t) = (x_t, \hat{v}_t)$ so that $\gamma(v_t, x_t, g_t) = [(x_t, g_t), (\hat{v}_t, a_t)]$ for some $a_t \in \mathcal{g}$, or alternatively $\gamma_{\alpha x_t}(v_t, g_t) = a_t$. Since $a_t = \dot{g}_t$ we have the differential equation $\gamma_{\alpha x_t}(v_t, g_t) = \dot{g}_t$ which can be solved to give us g_t provided g_0 is specified.

$$\begin{tikzcd}
 & U_\alpha \times V \arrow[dd, "\phi_\alpha"] & (x_t, \hat{v}_t, g_t, a_t) \\
\dot{\phi}_t & & \\
TE(U_\alpha, \mathcal{U}) \arrow[d, "\rho"] & TU_\alpha \times G \times \mathcal{g} \arrow[l, "\psi_{\alpha*}"'] \arrow[d, "(P,\mu_1)"] & \\
E(U_\alpha, \mathcal{U}) & U_\alpha \times G \arrow[l, "\psi_\alpha"'] & \\
\phi_t & (x_t, g_t) &
\end{tikzcd}$$

When g_0 is given, g_t is uniquely determined for each t. We can now ask what the condition is for γ to be a G-connection in the α-coordinate representation. If $\gamma_{\alpha x}(v,g) = g^{-1}\bar{\gamma}_{\alpha x}(gv)g$ where $\hat{v} = gv \in V$ for some function $\bar{\gamma}$ then the equation is $g_t\ \dot{g}_t\ g_t^{-1} = \bar{\gamma}_{\alpha x_t}(\hat{v}_t)$ and with $\hat{v}_t$ initially specified and determined, g_t has a left translation dependence on the initial choice of g_0. In approximating differential form, $g_{t+dt} - g_t = g_t\ \dot{g}_t\ dt = \bar{a}_t g_t dt$ viewed as maps $: V \to V$ where $\bar{a}_t = \bar{\gamma}_{\alpha x_t}(\hat{v}_t)$. From this we can obtain the left translation relation (if the limit exists) as

$$g_t = \lim_{n \to \infty} \left[\exp\left(\frac{t}{n}\,\bar{a}_{\frac{n-1}{n}t}\right) \exp\left(\frac{t}{n}\,\bar{a}_{\frac{n-2}{n}t}\right) \cdots \exp\left(\frac{t}{n}\,\bar{a}_0\right)\right] g_0$$

$$= \lim_{n \to \infty} \left(1 + \frac{t}{n}\,\bar{a}_{\frac{n-1}{n}t}\right)\left(1 + \frac{t}{n}\,\bar{a}_{\frac{n-2}{n}t}\right) \cdots \left(1 + \frac{t}{n}\,\bar{a}_0\right) g_0.$$

In the special case where $[\bar{a}_t, \bar{a}_s] = 0$ for all $t, s \in [0,1]$ we have $g_t = \exp\left(\int_0^t \bar{a}_s ds\right) g_0$. Of course $\bar{a}_s \in \mathcal{g}$ is viewed as a linear map from V to itself as is g_t, and 1 is e the identity map on V in the limit expression above.

(II.10) Transformation of Coordinates for a G-connection

A G-connection γ can be represented by a map $\bar{\gamma}_{\alpha x}: V \to \mathcal{g}$, and we may write $\bar{\gamma}_{\alpha x}(\hat{v}_\alpha) = \bar{a}_\alpha$ and $\bar{\gamma}_{\beta x}(\hat{v}_\beta) = \bar{a}_\beta$ where $\hat{v}_\beta = G_{\beta\alpha}(x)(\hat{v}_\alpha)$ and $\bar{a}_\alpha = g_\alpha a_\alpha g_\alpha^{-1}$, $\bar{a}_\beta = g_\beta a_\beta g_\beta^{-1}$. Then using (II.8.3),

$$a_\beta = g_\alpha^{-1} \tilde{G}_{\beta\alpha * x}(\hat{v}_\alpha) g_\alpha + a_\alpha, \quad \text{so}$$

$$g_\alpha a_\beta g_\alpha^{-1} = \tilde{G}_{\beta\alpha * x}(\hat{v}_\alpha) + \bar{a}_\alpha = g_\alpha g_\beta^{-1} \bar{a}_\beta g_\beta g_\alpha^{-1}.$$

Since $G_{\beta\alpha}(x) = g_\beta \cdot g_\alpha^{-1}$ we have

$$\bar{a}_\alpha = G_{\alpha\beta}(x) \bar{a}_\beta G_{\beta\alpha}(x) - \tilde{G}_{\beta\alpha * x}(\hat{v}_\alpha) \tag{II.10.1}$$

and this is the transformation of coordinate relation for the α-representation of the connection γ, when γ is a connection on the group structure.

Recall that in the definition of a connection, the fibre maps $\gamma_{\phi_{\alpha x}}: V \to E(M, \mathcal{U})_{\phi_{\alpha x}}$ were one to one, linear, with images $H_{\phi_{\alpha x}}$ which were horizontal subspaces. In the α-coordinate representation $\gamma_{\phi_{\alpha x}}(v) = \left(v, \gamma_{\alpha x}(v)\right)$ which is one-one and horizontal provided only that $\gamma_{\alpha x}$ (and hence $\bar{\gamma}_{\alpha x}$) is linear. The transformation equation can be written in the more convenient form

$$\bar{a}_\alpha = G_{\alpha\beta}(x)[\bar{a}_\beta + \tilde{G}_{\alpha\beta * x}(\hat{v}_\beta)] G_{\beta\alpha}(x). \tag{II.10.2}$$

(II.11) Components of a Connection

Let $v_{(1)},\cdots,v_{(n)}$ be a basis for V and $a_{(1)},\cdots,a_{(m)}$ a basis for $\mathcal{g}$. Since $\mathcal{g}$ consists of linear maps from V to V we have $n\times n$ matrices $A_{(1)},\cdots,A_{(m)}$ corresponding to $a_{(1)},\cdots,a_{(m)}$. This satisfies $a_{(i)}(v_{(j)}) = A_{(i)}{}^{k}{}_{j}\, v_{(k)}$ so that for scalars v^j, $a_{(i)}(v^j v_{(j)}) = v^j A_{(i)}{}^{k}{}_{j}\, v_{(k)}$ and the components of $v = v^j v_{(j)}$ are transformed by left multiplication by the matrix $A_{(i)}$.

There are structure constants $C_{i}{}^{k}{}_{j}$ which describe the Lie Algebra such that $[a_{(i)},a_{(j)}] = C_{i}{}^{k}{}_{j}a_{(k)}$, so $A_{(i)}A_{(j)} - A_{(j)}A_{(i)} = C_{i}{}^{k}{}_{j}A_{(k)}$. These constants depend upon the choice of the basis $a_{(1)},\cdots,a_{(m)}$ of $\mathcal{g}$.

A change of basis can be represented in the following simple form using primed and unprimed indices. We have $v_{(i')} = P^{i}_{i'}v_{(i)}$ and $v_{(i)} = P^{i'}_{i}\, v_{(i')}$ $i = 1,\cdots,n$ where $P^{i}_{i'}P^{i'}_{j} = \delta^{i}_{j}$ (inverses) and $P^{i'}_{j}\,P^{j}_{j'} = \delta^{i'}_{j'}$. Similarly $a_{(i')} = Q^{i}_{i'}a_{(i)}$, $i = 1,\cdots,m$, $a_{(i)} = Q^{i'}_{i}\,a_{(i')}$ where $Q^{i}_{i'}Q^{i'}_{j} = \delta^{i}_{j}$ and $Q^{i'}_{j}\,Q^{j}_{j'} = \delta^{i'}_{j'}$.

In order that $a_{(i')}(v_{(j')}) = A_{(i')}{}^{k'}{}_{j'}\, v_{(k')}$ we require that $A_{(i')}{}^{k'}{}_{j'} = Q^{i}_{i'}P^{k'}_{k}\,P^{j}_{j'}A_{(i)}{}^{k}{}_{j}$, and also we can easily see that $C_{i'}{}^{k'}{}_{j'} = Q^{k'}_{k}\,Q^{i}_{i'}Q^{j}_{j'}C_{i}{}^{k}{}_{j}$. Likewise $\overline{\gamma}_{\alpha x}: V \to \mathcal{g}$ can be represented by an $m\times n$ matrix $\overline{\Gamma}_{\alpha x}$. If $\overline{\gamma}_{\alpha x}(v) = a$ where $v = v^i v_{(i)}$ and $a = a^i a_{(i)}$ where $v^i,a^i \in \mathbb{R}$, $i = 1,2,\cdots,n$ (or m) then $\overline{\Gamma}_{\alpha x}{}^{i}{}_{j}v^j = a^i$. Under a change of basis it is clear that $a^{i'} = Q^{i'}_{i}\, a^i$ and $v^{i'} = P^{i'}_{i}\, v^i$ and so therefore $\overline{\Gamma}_{\alpha x}{}^{i'}{}_{j'} = Q^{i'}_{i}\, P^{j}_{j'}\overline{\Gamma}_{\alpha x}{}^{i}{}_{j}$. We define the connection symbol $\widetilde{\Gamma}_{\alpha x\, j}{}^{k}{}_{\ell} = A_{(i)}{}^{k}{}_{j}\, \overline{\Gamma}_{\alpha x}{}^{i}{}_{\ell}$. It transforms as $\widetilde{\Gamma}_{\alpha x\, j'}{}^{k'}{}_{\ell'} = P^{k'}_{k}\,P^{j}_{j'}P^{\ell}_{\ell'}\widetilde{\Gamma}_{\alpha x\, j}{}^{k}{}_{\ell}$, and furthermore if $v = v^i v_{(i)}$ then $\widetilde{\Gamma}_{\alpha x\, j}{}^{k}{}_{\ell}v^{\ell}$ represents in (kj) matrix form the image of v in $\mathcal{g}$ using the basis $v_{(i)}$ and viewed as a linear map from V to itself.

(II.12) <u>Change of Coordinates in Components</u>

Now for a fixed basis let us see how $\tilde{\Gamma}_{\alpha x}{}^{k}{}_{j\ell}$ transforms under a change of coordinates to $\tilde{\Gamma}_{\beta x}{}^{k}{}_{j\ell}$. Letting $v_{(i)}$, $i = 1,\cdots,n$ be the fixed basis, we have $\hat{v}_\alpha = \hat{v}_\alpha{}^{i} v_{(i)}$ and $\hat{v}_\beta = \hat{v}_\beta{}^{i} v_{(i)}$ where $\overline{\gamma}_{\alpha x}(\hat{v}_\alpha) = \overline{a}_\alpha$ and $\overline{\gamma}_{\beta x}(\hat{v}_\beta) = \overline{a}_\beta$. Also $\overline{a}_\alpha$ is associated with a matrix $\overline{A}_\alpha$ and $\overline{a}_\beta$ with another $n \times n$ matrix $\overline{A}_\beta$ using the fixed basis $\{v_{(i)}\}$ of V. Then $\overline{a}_\alpha(v_{(i)}) = \overline{A}_\alpha{}^{j}{}_{i} v_{(j)}$ and $\overline{a}_\beta(v_{(i)}) = \overline{A}_\beta{}^{j}{}_{i} v_{(j)}$ and $\tilde{\Gamma}_{\alpha x}{}^{k}{}_{j\ell} \hat{v}_\alpha^{\ell} = \overline{A}_\alpha{}^{k}{}_{j}$ and $\tilde{\Gamma}_{\beta x}{}^{k}{}_{j\ell} \hat{v}_\beta^{\ell} = \overline{A}_\beta{}^{k}{}_{j}$. We have $\hat{v}_\beta = G_{\beta\alpha}(x)(\hat{v}_\alpha)$, and with our basis $\{v_{(i)}\}$ of V we can view $G_{\beta\alpha}(x) \in G$ as a map $V \to V$ and represent it by a matrix that operates on components. Thus $\hat{v}_\beta^{i} = G_{\beta\alpha x}{}^{i}{}_{j} \hat{v}_\alpha^{j}$ and also $G_{\beta\alpha}(x)(v_{(j)}) = G_{\beta\alpha x}{}^{i}{}_{j} v_{(i)}$. Likewise the map $\tilde{G}_{\beta\alpha * x}: V \to \mathfrak{g}$ can be represented in component form as $G_{\beta\alpha * x}{}^{k}{}_{j\ell}$ exactly as $\overline{\gamma}_{\alpha x}$ is represented by $\tilde{\Gamma}_{\alpha x}{}^{k}{}_{j\ell}$. Then

$$\overline{a}_\alpha = G_{\alpha\beta}(x)[\overline{a}_\beta + \tilde{G}_{\alpha\beta * x}(\hat{v}_\beta)]G_{\beta\alpha}(x) \quad \text{from (II.10.2), and}$$

$$\underbrace{\overline{A}_\alpha{}^{i}{}_{p}}_{\tilde{\Gamma}_{\alpha x}{}^{i}{}_{pq}\hat{v}_\alpha^{q}} = G_{\alpha\beta x}{}^{i}{}_{k}[\underbrace{\overline{A}_\beta{}^{k}{}_{j}}_{\tilde{\Gamma}_{\beta x}{}^{k}{}_{j\ell}\hat{v}_\beta^{\ell}} + G_{\alpha\beta * x}{}^{k}{}_{j\ell}\underbrace{\hat{v}_\beta^{\ell}}_{G_{\beta\alpha x}{}^{\ell}{}_{q}\hat{v}_\alpha^{q}}]G_{\beta\alpha x}{}^{j}{}_{p}\,, \quad \text{so}$$

$$\tilde{\Gamma}_{\alpha x}{}^{i}{}_{pq}\hat{v}_\alpha^{q} = G_{\alpha\beta x}{}^{i}{}_{k}[\tilde{\Gamma}_{\beta x}{}^{k}{}_{j\ell} + G_{\alpha\beta * x}{}^{k}{}_{j\ell}]\,\hat{v}_\beta^{\ell}\, G_{\beta\alpha x}{}^{j}{}_{p}\,.$$

Since this is to hold for all $\hat{v}_\alpha^{q}$ we have the transformation relation for the connection components under a change of coordinates as

$$\tilde{\Gamma}_{\alpha x}{}^{i}{}_{pq} = G_{\alpha\beta x}{}^{i}{}_{k}[\tilde{\Gamma}_{\beta x}{}^{k}{}_{j\ell} + G_{\alpha\beta * x}{}^{k}{}_{j\ell}]G_{\beta\alpha x}{}^{\ell}{}_{q}\, G_{\beta\alpha x}{}^{j}{}_{p}\,. \tag{II.12.1}$$

Of course, we have the inverse relation $G_{\alpha\beta x}{}^{i}{}_{k}\, G_{\beta\alpha x}{}^{k}{}_{\ell} = \delta^{i}_{\ell}$.

(II.13) Homogeneous Bundle Charts

Because all the Lie Groups are subgroups of GL(V) and all Lie Algebras subalgebras of $\mathcal{gl}(V)$, every G-connection is a GL(V)-connection with the same components in the corresponding charts, with a wider range of charts and transforms corresponding to the bigger group GL(V). However the transformation relations, including the one above, are the same. Hence we can consider a special group of bundle charts (U_α, ϕ_α) which are *homogeneous*. If there exists a one-one smooth map $f_\alpha: U_\alpha \to V$ such that $f_{\alpha * x} = \phi_{\alpha x}^{-1}: M_x \to V$ for all $x \in U_\alpha$ then we say that (U_α, ϕ_α) is homogeneous.

$$\begin{tikzcd} TU_\alpha \arrow[r, "f_{\alpha*}"] \arrow[d, "P"'] & V \times V \arrow[d, "\text{first comp. proj.}"] \\ U_\alpha \arrow[r, "f_\alpha"] & V \end{tikzcd}$$

$f_{\alpha * x}: M_x \to V$

(is linear isomorphism)

If f_α exists it is not unique since $f_\alpha + v_0$, $v_0 \in V$ for fixed v_0 will also serve just as well. If (U_α, ϕ_α), (U_β, ϕ_β) are homogeneous charts with $x \in U_\alpha \cap U_\beta$, then $\phi_{\alpha x}^{-1} \circ \phi_{\beta x} = G_{\alpha\beta}(x) = (f_\alpha \circ f_\beta^{-1})_{*x}$. With the basis $\{v_{(i)}\}$ for V we have this expressed in matrix component form as

$$G_{\alpha\beta x}{}^{i}{}_{j} = \left.\frac{\partial f_\alpha{}^i}{\partial f_\beta{}^j}\right|_x$$ where we understand it is the components of the function $f_\alpha \circ f_\beta^{-1}: V \to V$ with respect to $\{v_{(i)}\}$ that are differentiated and evaluated at the point corresponding to $x \in U_\alpha \cap U_\beta$. Each f_α^i is a smooth function $f_\alpha^i: U_\alpha \to \mathbb{R}$ such that $f_\alpha = f_\alpha^i v_{(i)}$ and similarly $f_\beta = f_\beta^i v_{(i)}$. In general $f_\alpha \circ f_\beta^{-1}$ is not linear and is not defined on all of V but only on $f_\beta(U_\alpha \cap U_\beta)$. More explicitly, if

$$f_\alpha \circ f_\beta^{-1}(v^j v_{(j)}) = w^i v_{(i)} \quad \text{then} \quad \left.\frac{\partial f_\alpha{}^i}{\partial f_\beta{}^j}\right|_x = \left.\frac{\partial w^i}{\partial v^j}\right|_{v^k v_{(k)} = f_\beta(x)}.$$

Similarly, the map $\tilde{G}_{\alpha\beta * x}: V \to \mathcal{g}$ is represented in components by

$G_{\alpha\beta * x\,j\,\ell}{}^{k} = \left.\dfrac{\partial f_\beta{}^k}{\partial f_\alpha{}^i}\right|_x \left.\dfrac{\partial^2 f_\alpha{}^i}{\partial f_\beta{}^j \partial f_\beta{}^\ell}\right|_x$. In component form, the derivative map $\left.\dfrac{\partial^2 f_\alpha{}^k}{\partial f_\beta{}^j \partial f_\beta{}^\ell}\right|_x$ of $G_{\alpha\beta x\,j}{}^{k} = \dfrac{\partial f_\alpha{}^k}{\partial f_\beta{}^j}$ in the β representation is a Lie algebra element represented when contracted with some v^ℓ and left translated to the origin by $G_{\alpha\beta}(x)^{-1} = G_{\beta\alpha}(x)$, and this is the purpose of the $\left.\dfrac{\partial f_\beta{}^k}{\partial f_\alpha{}^i}\right|_x$ term. As a consequence, we determine the transformation law of the connection components with respect to a change of homogeneous coordinates as

$$
\begin{tikzcd}
(U_\alpha \cap U_\beta) \times V \arrow[d] \arrow[dr, "{(G_{\alpha\beta}, G_{\alpha\beta*})}"] & \\
T(U_\alpha \cap U_\beta) \arrow[r, "G_{\alpha\beta*}"'] \arrow[d, "P"] & G \times \mathfrak{g} \arrow[d, "\mu_1"] \\
U_\alpha \cap U_\beta \arrow[r, "G_{\alpha\beta}"'] & G
\end{tikzcd}
$$

$$\tilde{\Gamma}_{\alpha x\,p\,q}{}^{i} = \left(\tilde{\Gamma}_{\beta x\,j\,\ell}{}^{k} \left.\frac{\partial f_\alpha{}^i}{\partial f_\beta{}^k}\right|_x + \left.\frac{\partial^2 f_\alpha{}^i}{\partial f_\beta{}^j \partial f_\beta{}^\ell}\right|_x \right) \left.\frac{\partial f_\beta{}^\ell}{\partial f_\alpha{}^q}\right|_x \left.\frac{\partial f_\beta{}^j}{\partial f_\alpha{}^p}\right|_x .$$

If we then set $\Gamma_{\alpha x\,p\,q}{}^{i} = -\tilde{\Gamma}_{\alpha x\,p\,q}{}^{i}$ and $\Gamma_{\beta x\,j\,\ell}{}^{k} = -\tilde{\Gamma}_{\beta x\,j\,\ell}{}^{k}$ we see that Γ satisfies the usual familiar transformation law for components of a connection, namely

$$\Gamma_{\beta x\,j\,\ell}{}^{k} = \Gamma_{\alpha x\,p\,q}{}^{i} \left.\frac{\partial f_\beta{}^k}{\partial f_\alpha{}^i}\right|_x \left.\frac{\partial f_\alpha{}^q}{\partial f_\beta{}^\ell}\right|_x \left.\frac{\partial f_\alpha{}^p}{\partial f_\beta{}^j}\right|_x + \left.\frac{\partial f_\beta{}^k}{\partial f_\alpha{}^i}\right|_x \left.\frac{\partial^2 f_\alpha{}^i}{\partial f_\beta{}^j \partial f_\beta{}^\ell}\right|_x . \qquad \text{(II.13.1)}$$

(II.14) Asides and Comparisons

To give us confidence in the correctness of these results, and to relate them to more familiar properties of a connection, let us take the following digression on the transformation of $\dot{g}_t$ and the parallel transport formula.

(II.15) <u>The Transformation of</u> $\dot{g}_t$

Viewing g_t as a map $V \to V$ we have $\frac{d}{dt} g_t = g_t \dot{g}_t$ where $\dot{g}_t \in \mathfrak{g}$ and is the natural left translate representative of the derivative of g_t. For a coordinate change, $G_{\beta\alpha}(x_t) = g_{\beta t} \cdot g_{\alpha t}^{-1}$ so that $g_{\beta t} = G_{\beta\alpha}(x_t) g_{\alpha t}$. Differentiating these as linear maps from V to V we have

$$\frac{d}{dt} g_{\beta t} = \left[\frac{d}{dt} G_{\beta\alpha}(x_t)\right] g_{\alpha t} + G_{\beta\alpha}(x_t) \frac{d}{dt} g_{\alpha t}.$$

Now $\frac{d}{dt} G_{\beta\alpha}(x_t)$ is an element of $G_{G_{\beta\alpha}(x_t)}$ and is brought to a Lie algebra element by left translation by $G_{\alpha\beta}(x_t)$. From the diagram below, we see that the Lie algebra element is $\tilde{G}_{\beta\alpha * x_t}(\hat{v}_{\alpha t})$. Hence $\frac{d}{dt} G_{\beta\alpha}(x_t) = G_{\beta\alpha}(x_t) \tilde{G}_{\beta\alpha * x_t}(\hat{v}_{\alpha t})$.

$$\begin{tikzcd}
 & (U_\alpha \cap U_\beta) \times V \arrow[d, "\phi_\alpha"] \arrow[dr, "G_{\beta\alpha *} \circ \phi_\alpha"] & & \\
I \times \mathbb{R} \arrow[ur, "x_*(\alpha\text{-rep})"] \arrow[r, "x_*"'] \arrow[d, "\text{first comp.}"] & T(U_\alpha \cap U_\beta) \arrow[r, "G_{\beta\alpha *}"'] \arrow[d, "P"] & G \times \mathfrak{g} \arrow[d, "\mu_1"] \arrow[dr, "\mu_2"] & \\
I \arrow[r, "x"] \arrow[rr, bend right, "G_{\beta\alpha}(x_t),\ t \in I"'] & U_\alpha \cap U_\beta \arrow[r, "G_{\beta\alpha}"] & G & \mathfrak{g}
\end{tikzcd}$$

$(\dot{x} = x_*(1))$

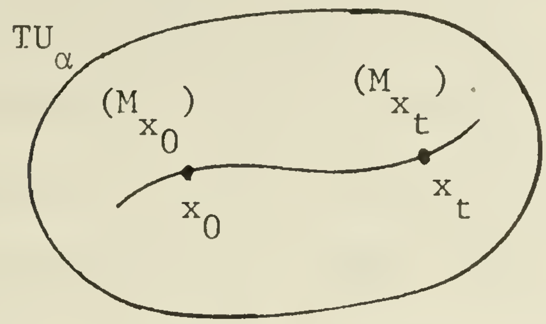
Now we can substitute this into the formula above for $\frac{d}{dt} g_{\beta t}$ and obtain,

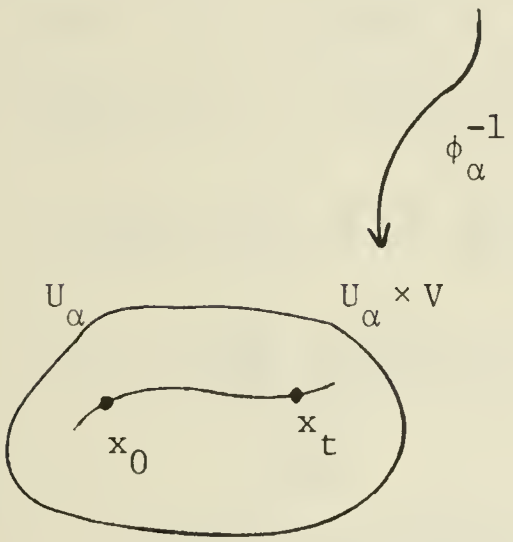
$g_{\beta t}\dot{g}_{\beta t} = G_{\beta\alpha}(x_t)\tilde{G}_{\beta\alpha * x_t}(\hat{v}_{\alpha t}) g_{\alpha t} + G_{\beta\alpha}(x_t) g_{\alpha t}\dot{g}_{\alpha t}$, and therefore

$\dot{g}_{\beta t} = g_{\alpha t}^{-1} \tilde{G}_{\beta\alpha * x_t}(\hat{v}_{\alpha t}) g_{\alpha t} + \dot{g}_{\alpha t}$ using $G_{\beta\alpha}(x_t) = g_{\beta t} \cdot g_{\alpha t}^{-1}$.

This transformation relation is the same one as we had between a_α and a_β and indeed it must be if the relation $a = \dot{g}$ we used is to be coordinate independent (c.f. (II.8.3)).

(II.16) The Parallel Transport Formula





We have seen that the parallel transport relation $\rho_t(\phi_{\alpha x_0}) = \phi_{\alpha_t x_t}$ mapping $\rho_t : L_{x_0}(\mathcal{U}_0) \to L_{x_t}(\mathcal{U}_0)$ satisfies $\psi^{-1}_{\alpha x_t} \circ \rho_t \circ \psi_{\alpha x_0} = L_{g_t g_0^{-1}}$. This allowed the introduction of another map, a linear isomorphism $\tilde{\rho}_t : M_{x_0} \to M_{x_t}$ defined by $\tilde{\rho}_t = \phi_{\alpha_t x_t} \circ \phi^{-1}_{\alpha x_0}$ (cf. II.6),

$$= [\rho_t(\phi_{\alpha x_0})] \circ \phi^{-1}_{\alpha x_0} .$$

Now $\rho_t = \psi_{\alpha x_t} \circ L_{g_t g_0^{-1}} \circ \psi^{-1}_{\alpha x_0}$ so that

$$\rho_t(\phi_{\alpha x_0}) = \psi_{\alpha x_t} \circ L_{g_t g_0^{-1}} \circ \underbrace{\psi^{-1}_{\alpha x_0}(\phi_{\alpha x_0})}_{\text{identity in } G} = \psi_{\alpha x_t}(g_t g_0^{-1}).$$

Hence $\tilde{\rho}_t = [\psi_{\alpha x_t}(g_t g_0^{-1})] \circ \phi^{-1}_{\alpha x_0}$. We represent the parallel transport $\tilde{\rho}_t$ in coordinates naturally through the map $\phi_{\alpha x_t} : V \to M_{x_t}$ for each t, and $\tilde{\rho}_t$ is thereby reduced to a linear isomorphism $: V \to V$ which can be reduced to a matrix using a basis $\{v_{(i)}\}$ for V. This map is $g_t g_0^{-1} : V \to V$ since $g_t g_0^{-1} = \phi^{-1}_{\alpha x_t} \circ \tilde{\rho}_t \circ \phi_{\alpha x_0}$. Also ρ_t itself can be looked upon through the $\psi_{\alpha x_t}$ and $\psi_{\alpha x_0}$ diffeomorphisms as a map $: G \to G$ which is equal to $L_{g_t g_0^{-1}}$. Hence we may write $g_t = g_t g_0^{-1} g_0$ and for fixed $g_t g_0^{-1}$ and variable g_0 we see the usual left translation relation reflected in the parallel transport.

(II.17) The Components of the Connection - Covariant Differentiation Condition

Let $w^i_t v_{(i)} \in V$ be represented by the components w^i_t of a vector field on the curve x, the value at x_t being $\phi_{\alpha x_t}(w^i_t v_{(i)})$. Let us

suppose that this field is parallel transport invariant. Then $0 = \frac{\delta}{dt}(w_t^{\ i}) = \frac{d}{dt} w_t^{\ i} + w_t^{\ k} \Gamma^{\ i}_{k\ j} \hat{v}_{\alpha t}^{\ j}$ where $\hat{v}_{\alpha t}^{\ j} = \phi^{-1}_{\alpha x_t}(\dot{x}_t)$. The transport condition is $w_t^{\ i} v_{(i)} = g_t g_0^{-1} w_0^{\ i} v_{(i)}$ or $w_t^{\ i} g_t^{-1} v_{(i)}$ is a constant independent of t. If $w_t = w_t^{\ i} v_{(i)}$ is the vector field in V representing the one on x, then $g_t^{-1} w_t$ = constant. Viewing g_t^{-1} as a map from V to V and w_t as an element of V we can differentiate with respect to t. We see that

$$0 = \frac{d}{dt}(g_t^{-1}) w_t + g_t^{-1} \frac{d}{dt} w_t,$$

$$0 = -g_t^{-1} \underbrace{\left(\frac{d}{dt} g_t\right)}_{g_t \dot{g}_t} g_t^{-1} w_t + g_t^{-1} \frac{d}{dt} w_t,$$

$$= -\dot{g}_t g_t^{-1} w_t + g_t^{-1} \frac{d}{dt} w_t, \Longrightarrow$$

$$\frac{d}{dt} w_t = g_t \dot{g}_t g_t^{-1} w_t$$

$$= \frac{d}{dt}(w_t^{\ i}) v_{(i)}$$

$$= -w_t^{\ k} \Gamma_{\alpha x_t\, k\ j}^{\ \ \ \ \ i} \hat{v}_{\alpha t}^{\ j} v_{(i)} \Longrightarrow$$

$$0 = \frac{d}{dt}(g_t^{-1} g_t)$$

$$0 = \left(\frac{d}{dt} g_t^{-1}\right) g_t + g_t^{-1} \frac{d}{dt} g_t$$

$$\Rightarrow \frac{d}{dt} g_t^{-1} = -g_t^{-1} \left(\frac{d}{dt} g_t\right) g_t^{-1}.$$

Also, $\frac{d}{dt} g_t = g_t \dot{g}_t$ where

$\dot{g}_t \in \mathfrak{g}$.

$w_t^{\ i} g_t \dot{g}_t g_t^{-1} v_{(i)} = -w_t^{\ k} \Gamma_{\alpha x_t\, k\ j}^{\ \ \ \ \ i} \hat{v}_{\alpha t}^{\ j} v_{(i)}$. Using $a_t = \dot{g}_t$ and $\bar{a}_t = g_t a_t g_t^{-1}$, we have

$$w_t^{\ i} \bar{a}_t v_{(i)} = -w_t^{\ k} \Gamma_{\alpha x_t\, k\ j}^{\ \ \ \ \ i} \hat{v}_{\alpha t}^{\ j} v_{(i)},$$

and in matrix form, using $\bar{a}_t(v_{(i)}) = \bar{A}_{t\ i}^{\ j} v_{(j)}$ we have

$$w_t^{\ i} \bar{A}_{t\ i}^{\ j} v_{(j)} = -w_t^{\ i} \Gamma_{\alpha x_t\, i\ k}^{\ \ \ \ \ j} \hat{v}_{\alpha t}^{\ k} v_{(j)}.$$

Since for any given fixed t we can make an arbitrary choice of $w_t^{\ i}$ (which will determine it for all t) we have $\bar{A}_{t\ i}^{\ j} = -\Gamma_{\alpha x_t\, i\ k}^{\ \ \ \ \ j} \hat{v}_{\alpha t}^{\ k}$. This gives us $\Gamma_{\alpha x\, i\ k}^{\ \ \ \ j} = -\tilde{\Gamma}_{\alpha x\, i\ k}^{\ \ \ \ j}$ by direct computation, since $\tilde{\Gamma}_{\alpha x\, j\ \ell}^{\ \ \ \ k} \hat{v}_{\alpha}^{\ \ell} = \bar{A}_{\alpha\ j}^{\ k}$ as we have seen.

(II.18) The Riemann Tensor

The linear map $\bar{\gamma}_{\alpha x}: V \to g$ is smoothly varying as a function of x and may itself be differentiated. We write it as $\bar{\gamma}_{\alpha}: U_{\alpha} \to L(V,g)$, and its derivative as a linear mapping $\dot{\bar{\gamma}}_{\alpha x}: V \to L(V,g)$. As the negative of the representative of $\bar{\gamma}_{\alpha x}$ in components is $\Gamma_{\alpha x\, i}{}^{j}{}_{k}$, so the negative of the representative of $\dot{\bar{\gamma}}_{\alpha x}$ is denoted by $\Gamma_{\alpha x\, i}{}^{j}{}_{k,\ell}$. If $v = v^{\ell} v_{(\ell)}$ then $\dot{\bar{\gamma}}(v)$ is represented by $-\Gamma_{\alpha x\, i}{}^{j}{}_{k,\ell} v^{\ell}$. The comma reduces to a partial derivative in the case of homogeneous coordinates.

$$\begin{tikzcd}
U_{\alpha} \times V \arrow[d, "\phi_{\alpha}"'] \arrow[drr, "(\bar{\gamma}_{\alpha}{,}\dot{\bar{\gamma}}_{\alpha})"] & & \\
TU_{\alpha} \arrow[rr, "\bar{\gamma}_{\alpha *}"'] \arrow[d, "P"'] & & L(V,g) \times L(V,g) \arrow[d, "\text{first component}"] \\
U_{\alpha} \arrow[rr, "\bar{\gamma}_{\alpha}"] & & L(V,g)
\end{tikzcd}$$

The comma generalizes the partial derivative, in fact

$G_{\alpha\beta x}{}^{i}{}_{k}\, G_{\alpha\beta * x}{}^{k}{}_{j\,\ell} = G_{\alpha\beta x}{}^{i}{}_{j,\ell}$ and in homogeneous coordinates, if they exist $G_{\alpha\beta x}{}^{i}{}_{j,\ell} = G_{\alpha\beta x}{}^{i}{}_{\ell,j}$. Observe that $G_{\alpha\beta x}{}^{i}{}_{j,\ell}$ is the directional derivative of $G_{\alpha\beta}(x)$ in the $V \to V$ representation with components given in terms of the fixed bases $\{v_{(i)}\}$ at the point $x \in U_{\alpha} \cap U_{\beta} \subset M$ along the vector direction $\phi_{\beta x}(v_{(\ell)})$ and proportional to its magnitude. The comma differentiation is understood as a matter of notation to be in the representation identified by the Greek letter closest to it, i.e. in $G_{\alpha\beta x}{}^{i}{}_{j,\ell}$ the differentiation is in the β-representation. For any v^{ℓ} we see that $G_{\alpha\beta x}{}^{i}{}_{j,\ell} v^{\ell}$ represents a map $: V \to V$ corresponding to an element of $G_{G_{\alpha\beta}(x)}$. We write $\dot{\bar{\gamma}}_{\alpha x}{}^{i}{}_{jk\ell} = \tilde{\Gamma}_{\alpha x\, j}{}^{i}{}_{k,\ell} = -\Gamma_{\alpha x\, j}{}^{i}{}_{k,\ell}$ and put $\bar{R}_{\alpha x}{}^{i}{}_{jk\ell} = \dot{\bar{\gamma}}_{\alpha x}{}^{i}{}_{jk\ell} - \dot{\bar{\gamma}}_{\alpha x}{}^{i}{}_{j\ell k} = 2\dot{\bar{\gamma}}_{\alpha x}{}^{i}{}_{j[k\ell]}$.

We can also introduce $\tilde{R}_{\alpha x}: V^{[2]} \to g$ which is linear and defined by the condition $\tilde{R}_{\alpha x}\left(v_{(i)} \wedge v_{(j)}\right) = [\bar{\gamma}_{\alpha x}(v_{(i)}), \bar{\gamma}_{\alpha x}(v_{(j)})]$ where $v_{(i)} \wedge v_{(j)} = \frac{1}{2}\left(v_{(i)} \otimes v_{(j)} - v_{(j)} \otimes v_{(i)}\right)$. An element $\omega \in V^{[2]}$ is

represented in components by ω^{ij} where $\omega^{ij} + \omega^{ji} = 0$ and $\omega = \omega^{ij} v_{(i)} \wedge v_{(j)}$, and $\omega = 2 \sum_{i<j} \omega^{ij} v_{(i)} \wedge v_{(j)}$. Hence $\tilde{R}_{\alpha x}(\omega) = \omega^{ij}[\bar{\gamma}_{\alpha x}(v_{(i)}), \bar{\gamma}_{\alpha x}(v_{(j)})]$. In component form $\tilde{R}_{\alpha x}{}^{i}{}_{jk\ell} = \Gamma_{\alpha x}{}^{i}{}_{pk}\Gamma_{\alpha x}{}^{p}{}_{j\ell} - \Gamma_{\alpha x}{}^{i}{}_{p\ell}\Gamma_{\alpha x}{}^{p}{}_{jk}$ and $\tilde{R}_{\alpha x}{}^{i}{}_{jk\ell}\omega^{k\ell}$ represents $\tilde{R}_{\alpha x}(\omega)$. Likewise the components $\bar{R}_{\alpha x}{}^{i}{}_{jk\ell}$ define a linear map $\bar{R}_{\alpha x}: V^{[2]} \to g$. We can then define $\hat{R}_{\alpha x}: V^{[2]} \to g$ by $\hat{R}_{\alpha x} = \bar{R}_{\alpha x} + \tilde{R}_{\alpha x}$. $\hat{R}_{\alpha x}$ is the *familiar Riemann tensor* in α-coordinate representation, and it has the components $\hat{R}_{\alpha x}{}^{i}{}_{jk\ell} = \Gamma_{\alpha x}{}^{i}{}_{j\ell,k} - \Gamma_{\alpha x}{}^{i}{}_{jk,\ell} + \tilde{R}_{\alpha x}{}^{i}{}_{jk\ell}$.

Notice that the transformation law for the connection symbols

$$\begin{aligned}\Gamma_{\beta x}{}^{k}{}_{j\ell} &= \Gamma_{\alpha x}{}^{i}{}_{pq} G_{\beta\alpha x}{}^{k}{}_{i} G_{\alpha\beta x}{}^{q}{}_{\ell} G_{\alpha\beta x}{}^{p}{}_{j} + G_{\alpha\beta * x}{}^{k}{}_{j\ell} \\ &= \Gamma_{\alpha x}{}^{i}{}_{pq} G_{\beta\alpha x}{}^{k}{}_{i} G_{\alpha\beta x}{}^{q}{}_{\ell} G_{\alpha\beta x}{}^{p}{}_{j} + G_{\beta\alpha x}{}^{k}{}_{i} G_{\alpha\beta x}{}^{i}{}_{j,\ell}\end{aligned}$$

will preserve symmetry of the connection symbols only for a transformation between homogeneous coordinates in general. Thus a symmetric connection is one with symmetric connection symbols in any (and every) coordinate system which is homogeneous. Such coordinates may not be consistent with the given group structure on the manifold M. Also, in structure preserving coordinates, a symmetric connection may have a non-symmetric connection symbol representation.

If we differentiate $G_{\alpha\beta x}{}^{i}{}_{j}$ in the α-representation to obtain $G_{\alpha\beta x}{}^{i}{}_{j\alpha,\ell}$ we can easily see that $G_{\alpha\beta x}{}^{i}{}_{j\alpha,\ell} = G_{\alpha\beta x}{}^{i}{}_{j,k} G_{\beta\alpha x}{}^{k}{}_{\ell}$.

The comma obeys the usual product rules of differentiation. We can best illustrate this by an example, rather than a proof. The equation $\tilde{G}_{\beta\alpha * x}(\hat{v}_{\alpha}) + G_{\alpha\beta}(x)\tilde{G}_{\alpha\beta * x}(\hat{v}_{\beta})G_{\beta\alpha}(x) = 0$ which we have already seen in (II.8.4) can be expressed as

$$G_{\beta\alpha}(x)\tilde{G}_{\beta\alpha * x}(\hat{v}_{\alpha}) + \tilde{G}_{\alpha\beta * x}(\hat{v}_{\beta})G_{\beta\alpha}(x) = 0,$$

and written in component form as

$$G_{\beta\alpha x}{}^{i}{}_{k} G_{\beta\alpha * x}{}^{k}{}_{j\,\ell}\, \hat{v}_{\alpha}{}^{\ell} + G_{\alpha\beta * x}{}^{i}{}_{k\,\ell_0}\, \hat{v}_{\beta}{}^{\ell_0} G_{\beta\alpha x}{}^{k}{}_{j} = 0$$

and since this holds for all components $\hat{v}_{\alpha}{}^{\ell}$ we see that

$$G_{\beta\alpha x}{}^{i}{}_{k} G_{\beta\alpha * x}{}^{k}{}_{j\,\ell} + G_{\alpha\beta * x}{}^{i}{}_{k\,\ell_0}\, G_{\beta\alpha x}{}^{\ell_0}{}_{\ell} G_{\beta\alpha x}{}^{k}{}_{j} = 0$$

and hence

$$G_{\beta\alpha x}{}^{i}{}_{k}(G_{\alpha\beta x}{}^{k}{}_{p} G_{\beta\alpha x}{}^{p}{}_{j,\ell}) + (G_{\beta\alpha x}{}^{i}{}_{p} G_{\alpha\beta x}{}^{p}{}_{k,\ell_0}) G_{\beta\alpha x}{}^{\ell_0}{}_{\ell} G_{\beta\alpha x}{}^{k}{}_{j} = 0\,,$$

$$G_{\beta\alpha x}{}^{i}{}_{j,\ell} + G_{\beta\alpha x}{}^{i}{}_{p} G_{\alpha\beta x}{}^{p}{}_{k\alpha,\ell} G_{\beta\alpha x}{}^{k}{}_{j} = 0,$$

$$G_{\alpha\beta x}{}^{p}{}_{i} G_{\beta\alpha x}{}^{i}{}_{j,\ell} + G_{\alpha\beta x}{}^{p}{}_{i\alpha,\ell} G_{\beta\alpha x}{}^{i}{}_{j} = 0,$$

$$(G_{\alpha\beta x}{}^{p}{}_{i} G_{\beta\alpha x}{}^{i}{}_{j})_{\alpha,\ell} = 0 = (\delta^{p}_{j})_{\alpha,\ell}\,.$$

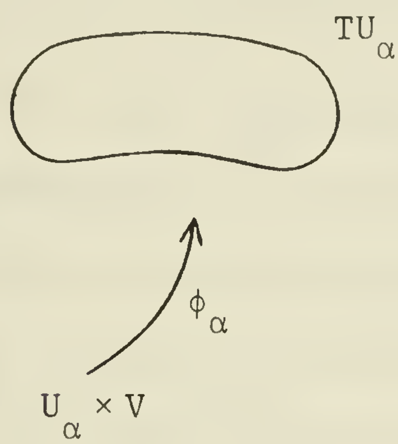
Notice that the differentiation must be consistently done in the α or β representation, and the extra α is added merely to indicate which representation.

(II.19) Covariant Differentiation

A tensor field on M or some neighborhood of a point x in M can be represented in the α bundle chart and the basis $\{v_{(i)}\}$ for V by components in the natural way. For a covariant (contravariant) vector field u_i (v^i) in the α representation we write the covariant derivatives as $u_{i|k} = u_{i,k} - u_j \Gamma^{j}{}_{i\,k}$ or $u_{\alpha i|k} = u_{\alpha i,k} - u_{\alpha j} \Gamma_{\alpha i}{}^{j}{}_{k}$ and $v_{\alpha}{}^{i}{}_{|k} = v_{\alpha}{}^{i}{}_{,k} + v_{\alpha}{}^{j} \Gamma_{\alpha j}{}^{i}{}_{k}$ where the x is dropped indicating each term is a function of x, at least in some local neighborhood. The components of these vector fields transform as $u_{\alpha i} = G_{\beta\alpha}{}^{j}{}_{i} u_{\beta j}$ and $v_{\alpha}{}^{i} = G_{\alpha\beta}{}^{i}{}_{j} v_{\beta}{}^{j}$. We can apply the comma differentiation to these expressions (remembering to stay in a fixed representation) and obtain the transformation law for $u_{\alpha i,k}$ and $v_{\alpha}{}^{i}{}_{,k}$ changing α to β.

Substituting into the formula for the covariant derivative and using the transformation law for the connection symbols we see that $u_{\alpha i|k} = G_{\beta\alpha}{}^{j}{}_{i} G_{\beta\alpha}{}^{\ell}{}_{k} u_{\beta j|\ell}$ and $v_{\alpha}{}^{i}{}_{|k} = G_{\alpha\beta}{}^{i}{}_{j} G_{\beta\alpha}{}^{\ell}{}_{k} v_{\beta}{}^{j}{}_{|\ell}$ so that these symbols transform as tensors.

(II.20) Coordinate Torsion Symbols



Let $v = v^i v_{(i)}$, $w = w^i v_{(i)}$, $u = u^i v_{(i)} \in V$. For each $x \in U_\alpha$, $\phi_{\alpha x}$ can be used to lift v, u, w to vectors in M_x so ϕ_α lifts u, v, w to vector fields U, V, W on U_α. If $(L_U V)(x) = -W(x) = [U,V](x)$ for some $x \in U_\alpha$ we write $w^i = T^{\alpha}_{x}{}^{i}_{j\ k} u^j v^k$ where $T^{\alpha}_{x}{}^{i}_{j\ k}$ is the *coordinate torsion* of the α system at x in component form. $T^{\alpha}{}^{i}_{j\ k}$ is understood to be a function of $x \in U_\alpha$ and $T^{\alpha}{}^{i}_{j\ k} = -T^{\alpha}{}^{i}_{k\ j}$. We write $T^{\alpha}{}_{\beta}{}^{i}_{j\ k}$ as the value of T^α in the β representation. For a homogeneous coordinate chart with map $\phi_\gamma^{-1} = f_{\gamma *}$ we have $T^\gamma = 0$. Clearly $T^{\alpha}{}_{\beta}{}^{i}_{j\ k} = T^{\alpha}{}^{p}_{\ell\ q} G_{\alpha\beta}{}^{\ell}{}_{j} G_{\alpha\beta}{}^{q}{}_{k} G_{\beta\alpha}{}^{i}{}_{p}$ changes the representation.

Let α, β be arbitrary representations and γ a homogeneous representation. Let v, w, u be as above where the components v^i, w^i, u^i as well as the basis $\{v_{(i)}\}$ are fixed. Let $V_{(\alpha)}$, $W_{(\alpha)}$, $U_{(\alpha)}$ be the lifted fields in the α-representation (each a vector field on U_α) and similarly for $V_{(\beta)}$, $V_{(\gamma)}$, $W_{(\beta)}$, $W_{(\gamma)}$, $U_{(\beta)}$, $U_{(\gamma)}$. Let $x \in U_\alpha \cap U_\beta \cap U_\gamma$ and suppose we consider only this neighborhood. Let $W_{(\alpha)\beta}{}^{i}$ be the i component of the vector field $W_{(\alpha)}$ in the β representation. This is clearly a function of x and its value at x may be denoted by $W_{(\alpha)\beta x}{}^{i}$. We can make similar definitions for other fields, and then clearly $v^i = V_{(\alpha)\alpha x}{}^{i}$ $\forall\, x \in U_\alpha \cap U_\beta \cap U_\gamma$ and also $W_{(\beta)\beta}{}^{i} \equiv w^i$, $U_{(\gamma)\gamma}{}^{i} \equiv u^i$,

$V_{(\beta)\beta}^{\ \ \ i} = v^i$ etc. Also we have the transformation relations between representations as $W_{(\alpha)\beta x}^{\ \ \ \ \ i} = G_{\beta\gamma x\ j}^{\ \ \ \ i} W_{(\alpha)\gamma x}^{\ \ \ \ \ j}$, or simply $W_{(\alpha)\beta}^{\ \ \ \ i} = G_{\beta\gamma\ j}^{\ \ \ i} W_{(\alpha)\gamma}^{\ \ \ \ j}$ as a function of x, and similarly for the other fields and representations.

Let $h: U_\alpha \to \mathbb{R}$ be a smooth function. We denote by $h_{\alpha,i}$ the i-derivative of h in the α-representation. Pick a curve x with $x(0) = x_0$ and $\dot{x}_0 = \phi_{\alpha x_0}(v_{(i)})$. Then $h_{\alpha,i}\Big|_{x_0} = \frac{d}{dt}(h \circ x)_t\Big|_{t=0}$. This value is independent of the choice of the curve x, and x is commonly taken to be the integral curve of the α-lift field of $v_{(i)}$ through x_0. Clearly the derivative transforms as $h_{\alpha,i} = G_{\beta\alpha\ i}^{\ \ \ j} h_{\beta,j}$ on $U_\alpha \cap U_\beta$. In a similar way we can differentiate a vector or tensor valued function h, or even transformation or connection symbol components in the α-representation. We denote the second derivative in components by $h_{\alpha,ij} = h_{\alpha,i,j}$. In homogeneous coordinates γ, $h_{\gamma,ij} = h_{\gamma,ji}$. Now

$$h_{\alpha,i} = G_{\gamma\alpha\ i}^{\ \ \ j} h_{\gamma,j} \quad \text{so} \quad h_{\alpha,ik} = G_{\gamma\alpha\ i,k}^{\ \ \ j} h_{\gamma,j} + \underbrace{G_{\gamma\alpha\ i}^{\ \ \ j} h_{\gamma,j\ell} G_{\gamma\alpha\ k}^{\ \ \ \ell}}_{\text{Symm. in i and k}}$$

and so

$$h_{\alpha,[ik]} = G_{\gamma\alpha\ [i,k]}^{\ \ \ j} h_{\gamma,j}.$$

Since the γ-representation is homogeneous, the Lie derivative takes the usual component form, i.e.

$$\mathcal{L}_{U_{(\alpha)}} V_{(\alpha)\gamma}^{\ \ \ \ i} = V_{(\alpha)\gamma,j}^{\ \ \ \ \ \ i} U_{(\alpha)\gamma}^{\ \ \ \ j} - V_{(\alpha)\gamma}^{\ \ \ \ j} U_{(\alpha)\gamma,j}^{\ \ \ \ \ \ i} .$$

Switching to α-representation we have,

$$G_{\gamma\alpha}{}^{i}{}_{k}\, \mathcal{L}_{U_{(\alpha)}} V_{(\alpha)}{}^{k} = V_{(\alpha)\gamma\alpha,j}{}^{i}\, U_{(\alpha)\alpha}{}^{j} - V_{(\alpha)\alpha}{}^{j}\, U_{(\alpha)\gamma\alpha,j}{}^{i} ,$$

and so at x,

$$-G_{\gamma\alpha}{}^{i}{}_{k}\, W^{k} = \left(V_{(\alpha)\alpha}{}^{k} G_{\gamma\alpha}{}^{i}{}_{k}\right)_{\alpha,j} U_{(\alpha)\alpha}{}^{j} - V_{(\alpha)\alpha}{}^{j} \left(U_{(\alpha)\alpha}{}^{k}\, G_{\gamma\alpha}{}^{i}{}_{k}\right)_{\alpha,j}$$

$$= v^{k} G_{\gamma\alpha}{}^{i}{}_{k,j} u^{j} - v^{j} u^{k} G_{\gamma\alpha}{}^{i}{}_{k,j}$$

$$= \left(G_{\gamma\alpha}{}^{i}{}_{k,j} - G_{\gamma\alpha}{}^{i}{}_{j,k}\right) u^{j} v^{k} = -2G_{\gamma\alpha}{}^{i}{}_{[j,k]} u^{j} v^{k} .$$

Hence $-W^{\ell} = -2G_{\alpha\gamma}{}^{\ell}{}_{i}\, G_{\gamma\alpha}{}^{i}{}_{[j,k]} u^{j} v^{k} \Rightarrow T^{\alpha}{}_{j}{}^{\ell}{}_{k} = 2G_{\alpha\gamma}{}^{\ell}{}_{i}\, G_{\gamma\alpha}{}^{i}{}_{[j,k]}$,

and this holds provided γ is a homogeneous representation.

For the derivative expression $h_{\alpha,i}$ above we can see that $h_{\alpha,[ik]} = \frac{1}{2} T^{\alpha}{}_{i}{}^{\ell}{}_{k}\, h_{\alpha,\ell}$ which is a result solely in the α-representation.

Notice that $G_{\alpha\beta}{}^{i}{}_{j} = G_{\alpha\gamma}{}^{i}{}_{k}\, G_{\gamma\beta}{}^{k}{}_{j}$ so that

$$G_{\alpha\beta}{}^{i}{}_{j,\ell} = G_{\alpha\gamma}{}^{i}{}_{k}\, G_{\gamma\beta}{}^{k}{}_{j,\ell} + G_{\alpha\gamma}{}^{i}{}_{k,p}\, G_{\gamma\beta}{}^{p}{}_{\ell}\, G_{\gamma\beta}{}^{k}{}_{j} ,$$

and

$$G_{\alpha\beta}{}^{i}{}_{[j,\ell]} = G_{\alpha\gamma}{}^{i}{}_{k}\, G_{\gamma\beta}{}^{k}{}_{[j,\ell]} + G_{\alpha\gamma}{}^{i}{}_{[k,p]}\, G_{\gamma\beta}{}^{p}{}_{\ell}\, G_{\gamma\beta}{}^{k}{}_{j} .$$

From $G_{\gamma\alpha}{}^{i}{}_{j}\, G_{\alpha\gamma}{}^{j}{}_{k} = \delta^{i}_{k}$ we obtain, taking α derivatives,

$$G_{\gamma\alpha}{}^{i}{}_{j,\ell}\, G_{\alpha\gamma}{}^{j}{}_{k} + G_{\gamma\alpha}{}^{i}{}_{j}\, G_{\alpha\gamma}{}^{j}{}_{k,p}\, G_{\gamma\alpha}{}^{p}{}_{\ell} = 0,$$

or

$$G_{\gamma\alpha}{}^{i}{}_{j,\ell} + G_{\gamma\alpha}{}^{k}{}_{j}\, G_{\gamma\alpha}{}^{i}{}_{q}\, G_{\alpha\gamma}{}^{q}{}_{k,p}\, G_{\gamma\alpha}{}^{p}{}_{\ell} = 0,$$

so that

$$G_{\gamma\alpha}{}^{i}{}_{[j,\ell]} + G_{\gamma\alpha}{}^{k}{}_{j}\, G_{\gamma\alpha}{}^{i}{}_{q}\, G_{\alpha\gamma}{}^{q}{}_{[k,p]}\, G_{\gamma\alpha}{}^{p}{}_{\ell} = 0.$$

This means that $G_{\alpha\gamma}{}^{i}{}_{[k,p]} = -\frac{1}{2} T^{\alpha}{}_{j}{}^{i}{}_{\ell}\, G_{\alpha\gamma}{}^{j}{}_{k}\, G_{\alpha\gamma}{}^{\ell}{}_{p}$ and hence

$$2G_{\alpha\beta}{}^{i}{}_{[j,\ell]} = G_{\alpha\beta}{}^{i}{}_{k}\, T^{\beta}{}^{k}{}_{j\,\ell} - T^{\alpha}{}^{i}{}_{p\,q}\, G_{\alpha\beta}{}^{p}{}_{j}\, G_{\alpha\beta}{}^{q}{}_{\ell} . \qquad \text{(II.20.1)}$$

Now $\Gamma_{\beta[j}{}^{k}{}_{\ell]} = \Gamma_{\alpha[p}{}^{i}{}_{q]}\, G_{\beta\alpha}{}^{k}{}_{i}\, G_{\alpha\beta}{}^{q}{}_{\ell}\, G_{\alpha\beta}{}^{p}{}_{j} + G_{\beta\alpha}{}^{k}{}_{i}\, G_{\alpha\beta}{}^{i}{}_{[j,\ell]}$, so $\Gamma_{\beta[j}{}^{k}{}_{\ell]} - \frac{1}{2} T^{\beta}{}^{k}{}_{j\,\ell}$ transforms like a tensor to $\Gamma_{\alpha[p}{}^{i}{}_{q]} - \frac{1}{2} T^{\alpha}{}^{i}{}_{p\,q}$. We write $T_{\beta\,j}{}^{k}{}_{\ell} = 2\Gamma_{\beta[j}{}^{k}{}_{\ell]} - T^{\beta}{}^{k}{}_{j\,\ell} = \Gamma_{\beta\,j}{}^{k}{}_{\ell} - \Gamma_{\beta\,\ell}{}^{k}{}_{j} - T^{\beta}{}^{k}{}_{j\,\ell}$ and call T_{β} the *torsion tensor* of the connection Γ in the β representation.

A simple evaluation of $A_{\alpha}{}^{a}{}_{|bc} - A_{\alpha}{}^{a}{}_{|cb}$ for an arbitrary vector field A_{α}^{a} in the α representation gives us

$$A_{\alpha}{}^{a}{}_{|bc} - A_{\alpha}{}^{a}{}_{|cb} = -A_{\alpha}{}^{a}{}_{|d}(\Gamma_{\alpha b}{}^{d}{}_{c} - \Gamma_{\alpha c}{}^{d}{}_{b}) + A_{\alpha}{}^{a}{}_{,bc} - A_{\alpha}{}^{a}{}_{,cb}$$
$$- A_{\alpha}{}^{e}\, \hat{R}_{\alpha}{}^{a}{}_{ebc} .$$

Using $A_{\alpha}{}^{a}{}_{,bc} - A_{\alpha}{}^{a}{}_{,cb} = A_{\alpha}{}^{a}{}_{,d}T^{\alpha}{}^{d}{}_{b\,c}$ we have the Ricci identity

$$A_{\alpha}{}^{a}{}_{|bc} - A_{\alpha}{}^{a}{}_{|cb} = -A_{\alpha}{}^{a}{}_{|d}\, T_{\alpha b}{}^{d}{}_{c} - A_{\alpha}{}^{e}\, R_{\alpha}{}^{a}{}_{ebc} \qquad \text{(II.20.2)}$$

where $R_{\alpha}{}^{a}{}_{ebc} = \hat{R}_{\alpha}{}^{a}{}_{ebc} + \Gamma_{\alpha e}{}^{a}{}_{d}\, T^{\alpha}{}^{d}{}_{b\,c}$, or

$$R_{\alpha}{}^{a}{}_{ebc} = \Gamma_{\alpha e}{}^{a}{}_{c,b} - \Gamma_{\alpha e}{}^{a}{}_{b,c} + \Gamma_{\alpha d}{}^{a}{}_{b}\Gamma_{\alpha e}{}^{d}{}_{c} - \Gamma_{\alpha d}{}^{a}{}_{c}\Gamma_{\alpha e}{}^{d}{}_{b} + \Gamma_{\alpha e}{}^{a}{}_{d}T^{\alpha}{}^{d}{}_{b\,c} . \qquad \text{(II.20.3)}$$

This modified form of the Riemann tensor with the coordinate torsion T^{α} added transforms like a tensor under a change of coordinates because the commutator equation for covariant differentiation is a tensor equation, and the usual quotient rule applies in this more general setting. It is clear that the usual familiar Riemann tensor is only a tensor with respect to homogeneous coordinate transforms. The new Riemann tensor can be looked upon as a map $R_{\alpha x}: V^{[2]} \to \mathcal{g}$ for each $x \in U_{\alpha}$, and the tensor transformation law has important implications with regard to the image of $R_{\alpha x}$ and holonomy groups and algebras.

We can also verify the following important equations:

$$B_{\alpha a|bc} - B_{\alpha a|cb} = B_{\alpha d}\, R_{\alpha}{}^{d}{}_{abc} - B_{\alpha a|e}\, T_{\alpha b}{}^{e}{}_{c} \tag{II.20.4}$$

where $B_{\alpha a}$ are the components of a smooth covector field in the α-representation,

$$h_{\alpha|bc} - h_{\alpha|cb} = -h_{\alpha,a}\, T_{\alpha b}{}^{a}{}_{c} \quad (h_{,a} = h_{|a}) \tag{II.20.5}$$

where h is a smooth scalar field. In such results as these when a single representation α is used throughout it may be dropped in order to simplify the notation. For instance, in the case of a second order contravariant tensor field γ^{ab} in components,

$$\gamma^{ab}{}_{|cd} - \gamma^{ab}{}_{|dc} = -\gamma^{ae}R^{b}{}_{ecd} - \gamma^{eb}R^{a}{}_{ecd} - \gamma^{ab}{}_{|e}T^{e}{}_{c\ d} \quad .$$

Because of the tensor transformation properties of $R_{\alpha x}: V^{[2]} \to \mathcal{g}$ it can be viewed as a map naturally lifted $R_x: M_x^{[2]} \to \mathcal{g}_x$, where G_x is the Lie group (of transformations on M_x) and $\mathcal{g}_x$ is the Lie Algebra. Allowing this to be a function of x we have $R: TM^{[2]} \to A(M,\mathcal{U})$ where $TM^{[2]} = \bigcup_{x\in M} M_x^{[2]}$ is the tensor bundle of second order contravariant skew tensors and $A(M,\mathcal{U})$ is the Lie Algebra bundle, i.e. $A(M,\mathcal{U}) = \bigcup_{x\in M} \mathcal{g}_x$ which is a vector bundle. The torsion tensor, written T_α with indices in component form, can likewise be viewed as a map $T: TM^{[2]} \to TM$ on the tensor bundles which is a smooth map. Equivalently $T_x \in TM^{1}_{[2]}$, $R_x \in TM^{1}{}_{1[2]}$ give us examples of the use of this notation. Obviously $TM = TM^1$ and $T^*M = TM_1$. To see the reasoning behind these equations, let us introduce some descriptive notation. Corresponding to $\phi_{\alpha x}: V \to M_x$ we define $\hat{\phi}_{\alpha x}: G \to G_x$ by $\hat{\phi}_{\alpha x}(g) = \phi_{\alpha x} \circ g \circ \phi_{\alpha x}^{-1}$ where $G \subset GL(V)$ and $G_x \subset GL(M_x)$ and these subsets are Lie subgroups, and $\hat{\phi}_{\alpha x}$ is a Lie group isomorphism. Define $\kappa_{\alpha\beta}: T(U_\beta \cap U_\alpha) \to T(U_\alpha)$ by

$\kappa_{\alpha\beta} = \phi_\alpha \circ \phi_\beta^{-1}$ and $\kappa_{\alpha x \beta y}: M_y \to M_x$ by $\kappa_{\alpha x \beta y} = \phi_{\alpha x} \circ \phi_{\beta y}^{-1}$. Let $\kappa_{\alpha\beta_x}$ denote $\kappa_{\alpha x \beta x}$ an element of G_x, so $\kappa_{\alpha\beta x} = \hat{\phi}_{\beta x}(G_{\beta\alpha}(x))$. The map $\hat{\phi}_{\alpha x}$ is actually a more general function defined on all of $L(V)$ with image in $L(V)$ the linear maps from V to itself. Consequently we also have that $\hat{\phi}_{\alpha x}: \mathcal{g} \to \mathcal{g}_x$ is a Lie algebra isomorphism. The $\hat{\phi}_\alpha: U_\alpha \times \mathcal{g} \to A(U_\alpha, \mathcal{U})$ are the bundle chart maps for the Lie algebra bundle $A(M,\mathcal{U})$. Furthermore $\hat{\phi}_{\alpha x}$ commutes with the exponential map exp which maps $\mathcal{g} \to G$ and $\mathcal{g}_x \to G_x$ so that $\hat{\phi}_{\alpha x} \circ \exp = \exp \circ \hat{\phi}_{\alpha x}: \mathcal{g} \to G_x$. Also $\hat{\phi}_{\beta x}^{-1} \circ \hat{\phi}_{\alpha x} = Ad_{G_{\beta\alpha}(x)}$ or $ad_{G_{\beta\alpha}(x)}$ for a domain of G or $\mathcal{g}$ respectively. In the Lie algebra bundle, the structure group G acts as a left translation on the fibre $\mathcal{g}$ through the map ad_g, $g \in G$. We can look upon ad_g as a left translation since $ad_{g_1} \circ ad_{g_2} = ad_{g_1 g_2}$. The coordinate transforms on the Lie algebra bundle between two bundle charts are $\hat{\phi}_{\beta x}^{-1} \circ \hat{\phi}_{\alpha x} = ad_{G_{\beta\alpha}(x)}$ which is "left translation" by the element $G_{\beta\alpha}(x)$ of G as in the case of the Lie tangent bundle and principal bundle. Of course $ad_g: \mathcal{g} \to \mathcal{g}$, for $g \in G$ defined by $ad_g(a) = gag^{-1}$ is a linear isomorphism from $\mathcal{g}$ to itself (i.e. an automorphism) for each $g \in G$. There exists a naturally defined linear map $tr: \mathcal{g} \to \mathbb{R}$ called the trace which is preserved by ad_g, i.e. $tr(ad_g(a)) = tr(a)$. Lie algebras obtained from Lie group structures whose groups are Lie subgroups of the unimodular or special linear groups will contain only elements of zero trace. In this case the contraction $R^i{}_{ijk}$ of the Riemann tensor for a structure preserving connection will be zero. We also have $e^{tr\, a} = \det(\exp(a))$ for $a \in \mathcal{g}$.

The adjoint operation on the Lie algebra $\mathcal{g}$ clearly commutes with the bracket $[ad_g a, ad_g b] = ad_g[a,b]$ indicating that ad_g is a Lie algebra automorphism. Clearly also $ad_{g^{-1}} = (ad_g)^{-1}$. We can see

that $\hat{\phi}_{\alpha x} \circ \hat{\phi}^{-1}_{\beta x} = Ad_{\kappa_{\alpha\beta x}}$ or more generally

$$\hat{\phi}_{\alpha x} \circ \hat{\phi}^{-1}_{\beta y} = Ad_{\kappa_{\alpha x\beta y}} , \quad \text{and} \quad \hat{\phi}_{\alpha x} \circ Ad_g \circ \hat{\phi}^{-1}_{\beta y} = Ad_{\kappa_{\alpha x g\beta y}}$$

where $g \in G$ and $\kappa_{\alpha x g\beta y} = \phi_{\alpha x} \circ g \circ \phi^{-1}_{\beta y}$.

We have developed a general formulation for the components of a tensor in an abstract (non-homogeneous) coordinate system. The laws of tensor manipulation and differentiation are very similar to those for the usual homogeneous coordinates and are thus easily understood both by theoretical physicists and mathematicians who work with differential geometry. For example, let us determine the formula for the Lie derivative of a vector field. Let X and Y be two vector fields on $U_\alpha \cap U_\gamma$. Assuming the γ system is homogeneous, we have

$$(\mathcal{L}_X Y)^i_\gamma = Y^i_{\gamma,j} X^j_\gamma - X^i_{\gamma,j} Y^j_\gamma \qquad \text{and so}$$

$$G_{\gamma\alpha}{}^i{}_k (\mathcal{L}_X Y)^k_\alpha = (Y_\alpha{}^k G_{\gamma\alpha}{}^i{}_k)_{,\ell} \underbrace{G_{\alpha\gamma}{}^\ell{}_j X^j_\gamma}_{X^\ell_\alpha} - (X_\alpha{}^k G_{\gamma\alpha}{}^i{}_k)_{,\ell} \overbrace{G_{\alpha\gamma}{}^\ell{}_j Y^j_\gamma}^{Y^\ell_\alpha}$$

$$= Y^k_{\alpha,\ell} X^\ell_\alpha G_{\gamma\alpha}{}^i{}_k + Y^k_\alpha X^\ell_\alpha G_{\gamma\alpha}{}^i{}_{k,\ell} - X^k_{\alpha,\ell} Y^\ell_\alpha G_{\gamma\alpha}{}^i{}_k - X^k_\alpha Y^\ell_\alpha G_{\gamma\alpha}{}^i{}_{k,\ell}$$

$$(\mathcal{L}_X Y)^k_\alpha = Y_\alpha{}^k{}_{,\ell} X^\ell_\alpha - X_\alpha{}^k{}_{,\ell} Y^\ell_\alpha + G_{\alpha\gamma}{}^k{}_i (G_{\gamma\alpha}{}^i{}_{j,\ell} - G_{\gamma\alpha}{}^i{}_{\ell,j}) Y^j_\alpha X^\ell_\alpha$$

$$= Y_\alpha{}^k{}_{,\ell} X^\ell_\alpha - X_\alpha{}^k{}_{,\ell} Y^\ell_\alpha + T^\alpha{}_j{}^k{}_\ell Y^j_\alpha X^\ell_\alpha ,$$

where T^α is the torsion of the α coordinate system. If a G-connection Γ is specified, we can express the Lie derivative as

$$(\mathcal{L}_X Y)^k_\alpha = Y_\alpha{}^k{}_{|\ell} X^\ell_\alpha - X_\alpha{}^k{}_{|\ell} Y^\ell_\alpha - T_{\alpha j}{}^k{}_\ell Y^j_\alpha X^\ell_\alpha$$

where $T_{\alpha j\ \ell}^{\ \ k} = -T^{\alpha\ k}_{\ j\ \ell} + \Gamma_{\alpha j\ \ell}^{\ \ k} - \Gamma_{\alpha\ell\ j}^{\ \ k}$ is the torsion tensor of the connection Γ and $|$ is its covariant derivative.

Since everything is done in the α representation we can drop the α for simplicity. For covariant vectors the form is

$$\begin{aligned} L_X W_a &= W_{a|b}X^b + W_b X^b{}_{|a} + T_{\alpha a\ c}^{\ \ b} X^c W_b \\ &= W_{a,b}X^b + W_b X^b{}_{,a} - W_b T^{\alpha\ b}_{\ a\ c} X^c . \end{aligned}$$

In general then, we can extend this to obtain

$$\begin{aligned} L_X R^{ab}{}_{cd} &= R^{ab}{}_{cd|e} X^e + R^{ab}{}_{ed}(X^e{}_{|c} + T_{\alpha c\ f}^{\ \ e} X^f) \\ &\qquad + R^{ab}{}_{ce}(X^e{}_{|d} + T_{\alpha d\ f}^{\ \ e} X^f) - R^{eb}{}_{cd}(X^a{}_{|e} + T_{\alpha e\ f}^{\ \ a} X^f) \\ &\qquad - R^{ae}{}_{cd}(X^b{}_{|e} + T_{\alpha e\ f}^{\ \ b} X^f) \\ &= R^{ab}{}_{cd,e} X^e + R^{ab}{}_{ed}(X^e{}_{,c} - T^{\alpha\ e}_{\ c\ f} X^f) \\ &\qquad + R^{ab}{}_{ce}(X^e{}_{,d} - T^{\alpha\ e}_{\ d\ f} X^f) - R^{eb}{}_{cd}(X^a{}_{,e} - T^{\alpha\ a}_{\ e\ f} X^f) \\ &\qquad - R^{ae}{}_{cd}(X^b{}_{,e} - T^{\alpha\ b}_{\ e\ f} X^f) \end{aligned}$$

for the case of a 2-contra-2-covariant tensor, and in the obvious way for more general tensors. Here $R^{ab}{}_{cd}$ is simply any tensor, not specifically the Riemann tensor.

(II.21) Examples: Galilei and Lorentz Structures and Comparisons

We can place Lorentz and Galilei group structures on a manifold M as a means of describing space-time, as well as a material connection on the body manifold B which is equipped with a group structure of material uniformity using the symmetry group of the material element, that is referred to by some authors as the isotropy group. If M is

a differential manifold which has a Lorentz metric g_{ab} defined everywhere, which is smooth, then g_{ab} induces the Lorentz group structure on M. Similarly if M has defined tensor fields γ^{ab} and u_c which are smooth, with $\gamma^{ab}u_b = 0$ and $\gamma^{ab}v_a v_b > 0$ for $v_a \neq \lambda u_a$ for any $\lambda \in \mathbb{R}$, then (γ^{ab},u_b) induces a Galilei group structure on M. If M has a Galilei group structure, $P: M \to B$ is the projection to the body B, and u^a is a field on M tangent to the flow we require $u^a u_a \neq 0$ everywhere, and normalize u^a so that $u^a u_a = -1$. Then there is a Lorentz metric $g^{ab} = \gamma^{ab} - u^a u^b$ defined on M, so M has a Lorentz group structure also. Conversely, if M has a Lorentz group structure and u^a is the flow vector $u^a u_a = -1$, then $\gamma^{ab} = g^{ab} + u^a u^b$ and $u_a = g_{ab}u^b$ can be used to construct a Galilei group structure (γ^{ab},u_b) on M. The relationship between these two structures will be considered at length ahead. It suffices to say for now that the Galilei structure approximates the Lorentz structure for velocities relative to the material rest frame that are much less than light, or equivalently, whose world lines are nearly tangent to the flow world lines. (Künzle [47]).

Let (M,g_{ab}) be a Lorentz space time with Lorentz group structure $\mathcal{G}(M) = (M,\mathcal{G},\mathcal{U},V,G)$ where G is L_V. For the remainder of this chapter let us work in (homogeneous) coordinates unless otherwise specified, returning to frame components (coordinates with torsion) in Chapter III. A connection $\Gamma_{j\ k}^{\ i}$ on the group structure $\mathcal{G}(M)$ is simply a metric connection, i.e. one satisfying $g_{ab|c} = 0$ or $K_{(ab)c} = 0$. The following results are easy to check.

<ins>Proposition</ins>: (a) *The manifold* M *is orientable if and only if* $\mathcal{G}(M)$ *has a subgroup structure with* $G = L_{V+}$.

(b) *The Lorentz space* (M,g_{ab}) *is time-sense preserving*

if and only if $G(M)$ *has a subgroup structure with* $G = L_V^{\uparrow}$.

The time-like vectors in M_x at x can be classified into two groups "future pointing" and "past pointing", and the notion of (M, g_{ab}) being time sense preserving merely means that the notion of future pointing (which in a sense is arbitrary since it is one of two classes in an equivalence relation) can be defined continuously everywhere on M.

In particular if M is that portion of space-time through which a material medium B moves with projection $P: M \to B$, then the time-like flow vector u^a defined on all of M singles out an equivalence class of time-like vectors at each point $x \in M$. (We say for $v, w \in V$, v and w are *equivalent* if $\langle Iv, w\rangle < 0$, (V, I) being an abstract Minkowski space. Since M_x also has the structure of an abstract Minkowski space, the notion above is well defined). Hence M is time sense preserving. Since B is orientable, we have seen that we can $(n+1=4)$ construct the everywhere non-zero 4-form $\hat{\eta}^*_{[ijk}u_{k]}$ on M so M is orientable. Therefore for the Lorentz space (M, g_{ab}) corresponding to a material body B, we find $G(M)$ has a subgroup structure with $G = L_{V+}^{\uparrow}$, and we can always assume the restricted or special Lorentz group is used in our group structures. Likewise for the corresponding Galilei structure of (M, γ^{ab}, u_b) we can assume that the proper Galilei group is the one under consideration when working with material media. If $G(M)$ is this Galilei group structure on M, how do we characterize a connection $\Gamma_b{}^a{}_c$ on this group structure? If $K_a{}^b{}_c$ is the contorsion of a covariant derivative $|$ whose connection is Galilei then $u_{a|b} = 0$ and $\gamma^{ab}{}_{|c} = 0$, and hence the contorsion satisfies

$$\begin{cases} K_{adc}u^d = -u_{a;c} \, , \\ (u_a u_b)_{;c} = K_{dac}\gamma^d_b + K_{dbc}\gamma^d_a \, , \quad \text{or equivalently} \end{cases} \tag{II.21.1}$$

$$\begin{cases} K_{(de)c}\gamma^e_a\gamma^d_b = 0 \\ K_{adc} = u_d \, u_{a;c} + \gamma^e_d K_{aec} \end{cases} \tag{II.21.2}$$

We call such a contorsion a *Galilei contorsion.*

The Galilei group is derived from the physical Newtonian transformation of frames just as the Lorentz group is derived from the Poincare transformation by differentiation. The general formula (in $\mathbb{R}^4$) for a change of frame (P.3) in classical space time is given by

$$\begin{bmatrix} x' \\ y' \\ z' \\ t' \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix}' = \left[\begin{array}{ccc|c} & Q(t) & & 0 \\ & & & 0 \\ \hline & & & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \left(\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} - \begin{bmatrix} x_0 \\ y_0 \\ z_0 \\ t_0 \end{bmatrix} \right) + \begin{bmatrix} c_1(t) \\ c_2(t) \\ c_3(t) \\ 0 \end{bmatrix} \tag{II.21.3}$$

where $Q(t)$ is an orthogonal matrix with determinant $+1$ (i.e. a rotation). The values of x_0, y_0, z_0 and t_0 are constants, $t' = t - t_0$. We then get by differentiating

$$\begin{bmatrix} dx \\ dy \\ dz \\ dt \end{bmatrix}' = \left[\begin{array}{ccc|c} & & & b_1 \\ & Q(t) & & b_2 \\ \hline & & & b_3 \\ 0 & 0 & 0 & 1 \end{array}\right] \begin{bmatrix} dx \\ dy \\ dz \\ dt \end{bmatrix} \quad \text{where} \quad \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = Q'(t) \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix} + \begin{bmatrix} c_1'(t) \\ c_2'(t) \\ c_3'(t) \end{bmatrix} .$$

The transformation (II.21.3) above satisfies $t' = t - t_0$ and for fixed t, $ds^2 = dx^2 + dy^2 + dz^2 = ds'^2$ so length is preserved. The (proper) Galilei group in $\mathbb{R}^4$ is the collection of all matrices below.

$$\left[\begin{array}{ccc|c} & & & b_1 \\ & Q & & b_2 \\ \hline & & & b_3 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Q is an arbitrary 3-rotation matrix and b_1, b_2, b_3 are arbitrary parameters. This matrix has determinant $+1$ and is the matrix exponential of the Galilei Lie

algebra matrix in (II.3.1), the b_1, b_2, b_3 parameters, of course, being different. The 3×3 orthogonal matrix Q is the exponential of

$$\begin{bmatrix} 0 & a_3 & -a_2 \\ -a_3 & 0 & a_1 \\ a_2 & -a_1 & 0 \end{bmatrix} = A, \qquad Q = \exp A = (\cos a)I - \left(\frac{\cos a - 1}{a^2}\right) \underset{\sim}{a}\, \underset{\sim}{a}^T + \frac{\sin a}{a} A$$

where I is the 3×3 identity and $\underset{\sim}{a}$ is the column vector $[a_1, a_2, a_3]^T$, $a^2 = a_1^2 + a_2^2 + a_3^2$, $a = \sqrt{a^2}$. Q is a 3-rotation about the axis of the direction $\underset{\sim}{a}$ through an angle equal to "a" according to the usual right hand rule.

The Galilei structure on a space-time is an important alternative to the Lorentz structure for approximating a description of space-time where all velocities of significance relative to the rest frame at each point are small compared to the speed of light. In this case the field equations of General Relativity can be significantly simplified, as we shall show.

Just for review, the proper orthochronous Lorentz group on $\mathbb{R}^4$ is the six dimensional Lie subgroup of $GL(\mathbb{R}^4)$ generated (in canonical matrix form) by the following three abelian 2-parameter Lie subgroups, for arbitrary values of the parameters [cf. (II.13.1)],

$$\exp\begin{bmatrix} 0 & 0 & 0 & b_1 \\ 0 & 0 & a_1 & 0 \\ 0 & -a_1 & 0 & 0 \\ b_1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \cosh b_1 & 0 & 0 & \sinh b_1 \\ 0 & \cos a_1 & \sin a_1 & 0 \\ 0 & -\sin a_1 & \cos a_1 & 0 \\ \sinh b_1 & 0 & 0 & \cosh b_1 \end{bmatrix} \tag{II.21.4}$$

$$\exp\begin{bmatrix} 0 & 0 & -a_2 & 0 \\ 0 & 0 & 0 & b_2 \\ a_2 & 0 & 0 & 0 \\ 0 & b_2 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \cos a_2 & 0 & -\sin a_2 & 0 \\ 0 & \cosh b_2 & 0 & \sinh b_2 \\ \sin a_2 & 0 & \cos a_2 & 0 \\ 0 & \sinh b_2 & 0 & \cosh b_2 \end{bmatrix} \tag{II.21.5}$$

$$\exp\begin{bmatrix} 0 & a_3 & 0 & 0 \\ -a_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_3 \\ 0 & 0 & b_3 & 0 \end{bmatrix} = \begin{bmatrix} \cos a_3 & \sin a_3 & 0 & 0 \\ -\sin a_3 & \cos a_3 & 0 & 0 \\ 0 & 0 & \cosh b_3 & \sinh b_3 \\ 0 & 0 & \sinh b_3 & \cosh b_3 \end{bmatrix} \quad \text{(II.21.6)}$$

The intersection of the Lorentz and Galilei groups on $\mathbb{R}^4$ is called the *rotation group* and is simply the 3 parameter rotation subgroup on the spacelike part of $\mathbb{R}^4$ (where $\mathbb{R}^4$ is equipped with the canonical Lorentz inner product). If M is a Lorentz space-time consisting of those world points through which a material medium B moves, then M, as we have seen, has a Galilei structure which is the material rest frame approximation to its Lorentz structure. The pointwise intersection of the groups in the Lorentz and Galilei group structures give us the rotation group structure. We can then talk about *rotation connections*(connections on the rotation structure) and the corresponding covariant differentiation and contorsion. A connection is clearly a rotation connection if and only if it is both metric (Lorentz) and Galilei if and only if $g_{ab|c} = 0$ and $u^a{}_{|b} = 0$ for the corresponding covariant differentiation. The star, dot and fundamental connections are therefore rotation[1] connections (P.2). As in (II.1) the Lie tangent and principle bundles and in (II.20) the Lie algebra bundle can be determined for the Lorentz, Galilei and rotation group structures, as well as for the symmetry group structure on B as in Wang [108], p. 46-62.

(II.22) Kinematics in the Galilei Structure

One of the most important properties of Galilei connections is that the covariant derivative of the metric volume element is always zero, i.e.

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[1] A connection is rotation if and only if it is metric and flow constant.

$\varepsilon^{abcd}{}_{|e} = 0 = \varepsilon_{abcd|e}$ for $|$ a Galilei differentiation. To prove this we observe that $\varepsilon^{1234}{}_{|e} = -\varepsilon^{1234} K_a{}^a{}_e$ and $\varepsilon_{1234|e} = \varepsilon_{1234} K_a{}^a{}_e$ and then $K_a{}^a{}_e = 0$ from (II.21.2). In fact the Galilei structure (γ^{ab}, u_a) on M defines for us (sign handled by orientation) a volume element without specifically mentioning the flow u^a. Then for any smooth "flow vector" u^a with $u^a u_a = -1$ on M, the corresponding Lorentz metric $g^{ab} = \gamma^{ab} - u^a u^b$ has the same volume element, independent of u^a, which is also the Galilei volume element. This comes from the fact that every element of the proper Galilei group on $\mathbb{R}^4$ has determinant $+1$.

The next important result to show is that the deformation of a material medium is the same, whether measured in a Lorentz space M or the Galilei structure approximating this for flow rest frames.

Suppose that M is equipped with a Galilei structure (whose tensors are) (γ^{ab}, u_a), with body B, world lines $P^{-1}(X)$, $X \in B$. If $x \in P^{-1}(X)$, let $M_x^\perp$ be the tangent subspace of M_x orthogonal to u_a at x. Then $\gamma^{ab}|_x \in \mathrm{Sym}^+(M_x^{\perp *}, M_x^\perp)$ and putting $p_x = P_{*x}|M_x^\perp$ which is an isomorphism $: M_x^\perp \to B_X$ we have the natural Galilei deformation $g_{\alpha\beta}(\tau)$ at X corresponding to x as $(p \circ \gamma \circ p^*)^{-1} \in \mathrm{Sym}^+(B_X, B_X^*)$. This is, as we have seen, precisely the same as the usual Lorentz deformation at (X,τ). Hence, for kinematical description, it does not matter whether we use Lorentz or Galilei structure invariant tensors to measure deformation.

(II.23) Galilei Connections

There are particular types of Galilei connections (which need not be metric) that are worth considering here, on a space-time with a material medium. Since the parallel transports of a Galilei connection

map $M_x^\perp$ isomorphically and isometrically onto $M_{x'}^\perp$, for $x, x' \in c$, a curve, we can naturally introduce the notion of a restricted class of Galilei connections known as *material Galilei connections*.

A Galilei connection is *material* if the induced parallel transports on the orthogonal spaces $M_x^\perp$ for $x \in c$ are fundamental transports for every curve c. Thus it is assumed, as in the case of the fundamental connection (I.21), that a material connection is provided on the smooth materially uniform simple body B.

Let $\overset{\vee}{K}_{bac}$ denote a material Galilei contorsion (with material Galilei derivative $\vee$ so $u_{a\vee b} = 0$ and $\gamma^{ab}{}_{\vee c} = 0$ in particular), and let K_{bac} denote the fundamental contorsion with fundamental covariant derivative $|$ (I.21). Then recall that $K_{bac} = \hat{K}_{[ba]c}$ where $\hat{K}_{bac}$ is the material contorsion. The condition for fundamental transport of orthogonal vectors along any curve c by the contorsion $\overset{\vee}{K}_{bac}$ is easily seen to be $\gamma_d^b \overset{\vee}{K}_{bac} = \gamma_d^b K_{bac}$. A material Galilei connection is not unique since we have not specified how vectors with a component parallel to $\underset{\sim}{u}$ will transport along c. In terms of the delta tensor (I.22)

$$\begin{aligned}\gamma_d^b \overset{\vee}{K}_{bac} &= \Delta_{dac} + u_{d;c} u_a + u_c (u_{[d;a]} + \dot{u}_{[d} u_{a]}) \\ &= \Delta_{dac} + u_{d;c} u_a + u_c \omega_{da} . \qquad \text{(II.23.1)}\end{aligned}$$

It is clear that this satisfies (II.21.2), i.e. it is consistent with $\overset{\vee}{K}_{adc} u^d = -u_{a;c}$ and gives us $\gamma_e^a \gamma_d^b \overset{\vee}{K}_{bac} = \Delta_{dec} + u_c \omega_{de}$. Moreover complete orthogonalization gives $\gamma_f^c \gamma_e^a \gamma_d^b \overset{\vee}{K}_{bac} = \Delta_{def}$ which recovers the delta tensor.

Since we have determined $\gamma_d^b \overset{\vee}{K}_{bac}$, the contorsion is completely

known if we determine $u^{b\vee}K_{bac} = u_{a;c} - u_{\downarrow a\vee c}$. (Notice that $u_{a\vee b} = 0$, $u_{\downarrow a\vee c} = g_{ab}u^{b}{}_{\vee c} \neq 0$). If we take $u^{b}{}_{\vee c} = 0$ then $\vee$ simply becomes the fundamental covariant derivative (which is rotation - hence Galilei and Lorentz). $u_{\downarrow a\vee b}$ is subject to the condition $u^{a}u_{\downarrow a\vee b} = 0$, since clearly $u_{a}u^{a}{}_{\vee b} = 0$.

We recall that a connection with contorsion K_{abc} is spatially nonrotating along arbitrary curves in space-time (see the end of section (I.27)) if for some w_{ac}, v_{bc}, $K_{abc} = u_{b}w_{ac} + u_{a}v_{bc}$. If it is also to be Galilei, we impose (I.21.2), and the condition is $w_{ac} = u_{a;c} + u_{a}(u^{b}v_{bc})$ which is necessary and sufficient. The simplest case, taking $v_{bc} = 0$ giving $K_{abc} = u_{b}u_{a;c}$ is called the *basic Galilei contorsion*. This will be of great importance in the following sections.

(II.24) <u>Symmetric Galilei Connections</u>

A Galilei connection is symmetric if its contorsion satisfies $K_{a\ c}^{\ b} = K_{c\ a}^{\ b}$. Because of (II.21.1) such connections exist if and only if $\dot{u}^{a} = 0$ and $\omega_{ab} = 0$, i.e. the motion is geodesic and irrotational, i.e. $u_{a,b} = u_{b,a}$, or $du_{a|b} = 0$ or $u_{a;b} = \theta_{ab}$. When these connections do exist, they are not unique, as in the case of symmetric Lorentz or Riemann connections, where one unique connection of this type always exists.

For a symmetric Galilei connection, $K_{abc}\gamma^{a}_{d}\gamma^{b}_{e}\gamma^{c}_{f}$ is symmetric in d and f and antisymmetric (II.21.2) in d and e and is therefore zero. Hence $K_{aec}\gamma^{a}_{b}\gamma^{e}_{d} = \Lambda_{db}u_{c}$ where Λ_{db} is antisymmetric and orthogonal. Now $\gamma^{a}_{b}K_{adc} = u_{d}u_{b;c} + \Lambda_{db}u_{c}$ from (II.21.1) and therefore

$$K_{adc} = u_{d}u_{a;c} + \Lambda_{da}u_{c} + \Lambda_{dc}u_{a} + t_{d}u_{a}u_{c} \qquad \text{(II.24.1)}$$

where $t_{d}u^{d} = 0$, for some t_{d}, gives us the general solution. Of

course it is assumed in this section that a symmetric Galilei connection exists, i.e. $u_{a;b} = \theta_{ab}$. Under these conditions, the basic Galilei connection ((II.23), last part) is always symmetric. Unlike some of the earlier connections we introduced, the symmetric Galilei connection is intended to be an alternate (approximate) description of the dynamical properties of space-time (Papapetrou [81], pp. 55-59) and for this reason we choose it to be symmetric. If it is to describe the space-time appropriately, acting as a low velocity approximation to the Lorentz space-time in rest frames, it should be *spatially nonrotating* (II.23, last part). In (II.24.1) we are required to take the spatially non-rotating Galilei contorsion

$$K_{abc} = u_b u_{a;c} + u_b u_a u^d v_{dc} + u_a v_{bc} \qquad \text{(II.24.2)}$$

and comparison gives $v_{ab} = t_a u_b - u_a v_b$ so

$$K_{bac} = u_a u_{b;c} + u_b u_c t_a \qquad \text{(II.24.3)}$$

for the spatially nonrotating symmetric Galilei contorsion. Dynamical considerations later on will have us take $t_a = 0$ for the basic contorsion to be considered.

The antisymmetric orthogonal tensor Λ_{da} and orthogonal vector t_d in (II.24.1) can be combined in a unique way to give an anti-symmetric tensor $\overset{\circ}{\Lambda}_{da}$. This is

$$\Lambda_{da} u_c + \Lambda_{dc} u_a + t_d u_a u_c = \gamma^e_d \overset{\circ}{\Lambda}_{ea} u_c + \gamma^e_d \overset{\circ}{\Lambda}_{ec} u_a$$

or, more simply,

$$\overset{\circ}{\Lambda}_{da} = \Lambda_{da} + u_{[a} t_{d]} . \qquad \text{(II.24.4)}$$

This will be used later in our study of Newtonian connections. In

particular, $2\gamma^{e\circ}_{d}\Lambda_{e(a}u_{c)}$ is the difference between the symmetric Galilei connection (II.24.1) and the basic symmetric Galilei connection. The difference between any two symmetric Galilei connections (or contorsions) is also of this form. (cf. Künzle [46], p. 349).

The Galilei structure is made to approximate the Lorentz structure so that for low velocities we can have a transition from relativistic to Newtonian mechanics. However if we take a Galilei structure defining orthogonal spaces $M^{\perp}_{x}$ representative of time tangent frames[1] to the flow lines of a material medium, it may, in the most general cases <u>not</u> be *integrable* i.e. $du_{a}\big|_{b} \neq 0$.

A Galilei structure $(\tilde{\gamma}^{ab}, \tilde{u}_{a})$ on M is said to be *compatible* with the metric g_{ab} if we let $\tilde{u}^{a} = g^{ab}\tilde{u}_{b}$ then $g^{ab} = \tilde{\gamma}^{ab} - \tilde{u}^{a}\tilde{u}^{b}$. Clearly, given any metric tensor field g_{ab} on M (of Lorentz type) and any 1-form $\tilde{u}_{a}$ with $d\tilde{u}_{a}\big|_{b} = 0$ and $g^{ab}\tilde{u}_{a}\tilde{u}_{b} = -1$ we can find a compatible integrable Galilei structure simply by taking $\tilde{\gamma}^{ab} = g^{ab} + \tilde{u}^{a}\tilde{u}^{b}$, where $\tilde{u}^{a} = g^{ab}\tilde{u}_{b}$. A symmetric Galilei contorsion (consistent with this Galilei structure) is of the form

$$K_{adc} = \tilde{u}_{d}\tilde{u}_{a;c} + \Lambda_{da}\tilde{u}_{c} + \Lambda_{dc}\tilde{u}_{a} + t_{d}\tilde{u}_{a}\tilde{u}_{c} \qquad \text{(II.24.4)}$$

where $\tilde{u}_{a;c}$ is symmetric in a and c. It satisfies the relations

$$\begin{aligned} &K_{(de)c}\tilde{\gamma}^{e}_{a}\tilde{\gamma}^{d}_{b} = 0 \qquad (\tilde{\gamma}^{e}_{a} = \delta^{e}_{a} + \tilde{u}^{e}\tilde{u}_{a}) \\ &K_{adc} = \tilde{u}_{d}\tilde{u}_{a;c} + \tilde{\gamma}^{e}_{d}K_{aec} \end{aligned} \qquad \text{(II.24.5)}$$

which characterize a $(\tilde{\gamma}^{ab}, \tilde{u}_{a})$-Galilei connection, as well as the symmetry relation $K_{dec} = K_{ced}$. Λ_{da} is antisymmetric and both it and

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[1] A frame is *time tangent* to a time-like normal field $\tilde{u}^{a}$, if $\tilde{u}^{a}$ is the time-like vector for the frame under consideration.

t_d are orthogonal to $\tilde{u}^a$ in (II.24.4).

(II.25) The Riemann Tensor and Torsion for Galilei Connections

Suppose $u_{d;c} = \theta_{dc}$ and $K_d{}^a{}_c = u^a u_{d;c}$ is the basic (symmetric) Galilei contorsion. The associated basic Galilei Riemann tensor $R^a{}_{dbc} = \mathcal{R}^a{}_{dbc} - \mathcal{C}^a{}_{dbc}$ is given (I.17) by

$$\mathcal{C}^a{}_{dbc} = K_d{}^a{}_{c;b} - K_d{}^a{}_{b;c} + K_d{}^f{}_b K_f{}^a{}_c - K_d{}^f{}_c K_f{}^a{}_b \ . \qquad \text{(II.25.1)}$$

Using $K_d{}^a{}_c = u^a u_{d;c}$ where $u_{d;c} = u_{c;d}$ is orthogonal, we have

$$\begin{aligned}\mathcal{C}_{adbc} &= u_{a;b}u_{d;c} - u_{a;c}u_{d;b} + u_a u_{d;cb} - u_a u_{d;bc} \\ &= u_{a;b}u_{d;c} - u_{a;c}u_{d;b} - u_a \mathcal{R}^e{}_{dbc}u_e \ ,\end{aligned} \qquad \text{(II.25.2)}$$

where $\mathcal{R}^e{}_{dbc}$ is the Christoffel symbol Riemann tensor (for the Levi-Civita connection). Clearly $\mathcal{C}_{a[dbc]} = 0$ so $R_{a[dbc]} = 0$, and $\mathcal{C}^a{}_{adb} = 0$ so $R^a{}_{adb} = 0$. Contracting the cyclic identity $R_{a[dbc]} = 0$ we get $R_{db} = R_{bd}$ where $R_{db} = R^a{}_{dba}$ is the Ricci tensor, and the Lorentz metric is used to raise indices. We may also write $R_{db} = \mathcal{R}_{db} - \mathcal{C}_{db}$ where $\mathcal{C}_{db} = \mathcal{C}^a{}_{dba} = -K_d{}^a{}_{b;a} = -(u^a u_{d;b})_{;a}$ where, of course, $u_{d;b} = \theta_{db}$.

The symmetry results illustrated here for the basic Galilei connection on the Riemann tensor hold in a more general setting. Recall that for a connection on a group structure with torsion tensor $T_b{}^d{}_c$ we may write in torsion-free homogeneous frame components (coordinates),

$$\begin{aligned}R^a{}_{dbc} &= \Gamma_d{}^a{}_{c,b} - \Gamma_d{}^a{}_{b,c} + \Gamma_d{}^f{}_c \Gamma_f{}^a{}_b - \Gamma_d{}^f{}_b \Gamma_f{}^a{}_c \\ A^a{}_{|bc} - A^a{}_{|cb} &= -R^a{}_{dbc}A^d - A^a{}_{|d}T_b{}^d{}_c \\ B_{a|bc} - B_{a|cb} &= R^d{}_{abc}B_d - B_{a|d}T_b{}^d{}_c \\ \phi_{|bc} - \phi_{|cb} &= -\phi_{,d}T_b{}^d{}_c \\ P^{ab}{}_{|cd} - P^{ab}{}_{|dc} &= -P^{ae}R^b{}_{ecd} - P^{eb}R^a{}_{ecd} - P^{ab}{}_{|e}T_c{}^e{}_d\end{aligned} \qquad \text{(II.25.3)}$$

where $T_{b\ c}^{\ a} = \Gamma_{b\ c}^{\ a} - \Gamma_{c\ b}^{\ a}$ and these equations are the Ricci identities. Now if (γ^{ab}, u_a) is a Galilei structure on M and $\Gamma_{b\ c}^{\ a}$ is a Galilei connection, $|$ is its covariant derivative and $R^a_{\ bcd}$ is its Riemann tensor, then $\gamma^{ab}{}_{|c} = 0$ and $u_{d|c} = 0$ so we get (putting $P^{ab} = \gamma^{ab}$)

$$u_a R^a_{\ bcd} = 0, \quad \gamma^{ae} R^b_{\ ecd} + \gamma^{be} R^a_{\ ecd} = 0. \tag{II.25.4}$$

For any connection which is symmetric on M we can prove the *Cyclic* and *Bianchi* identities, namely

$$R^a_{\ [bcd]} = 0, \quad R^a_{\ b[cd|e]} = 0 \tag{II.25.5}$$

and this will be done more generally (for non-zero torsion) in Chapter III. Thus we can see that the Ricci tensor R_{ab} for a symmetric Galilei connection is symmetric. In fact, from the cyclic identity this is true if and only if $R^a_{\ abc} = 0$, and from (II.25.1),

$$C^a_{\ abc} = K_{a\ c;b}^{\ a} - K_{a\ b;c}^{\ a} + K_{a\ b}^{\ f} K_{f\ c}^{\ a} - K_{a\ c}^{\ f} K_{f\ b}^{\ a} = 0$$

since the last two terms cancel, and $K_{a\ c}^{\ a} = g^{ad} K_{adc} = 0$ directly from (II.24.1).

(II.26) Comparison of Lorentz Structures Compatible with a Galilei Structure

Let (γ^{ab}, u_a) be an integrable Galilei structure $(du_a|_b = 0)$ and let u^a be a vector field on M with $u_a u^a = -1$. Let v^a be any smooth field with $v^a u_a = 0$ everywhere on M and let $u'^a = u^a + v^a$. Define $g^{ab} = \gamma^{ab} - u^a u^b$ and $g'^{ab} = \gamma^{ab} - u'^a u'^b$ and let g_{ab}, g'_{ab} be the corresponding inverses, $\left\{{a \atop b\ c}\right\}$, $\left\{{a \atop b\ c}\right\}'$ the corresponding Christoffel symbols and $\Gamma_{b\ c}^{\ a} = \left\{{a \atop b\ c}\right\} - u^a u_{b;c}$ and $\Gamma'{}_{b\ c}^{\ a} = \left\{{a \atop b\ c}\right\}' - u'^a u_{b;'c}$ the corresponding basic Galilei connections. Then

$g'^{ab} = g^{ab} - v^a u^b - u^a v^b - v^a v^b$ and writing $g'_{ab} = g_{ab} + h_{ab}$ we find $h_{ac} = v_a u_c + u_a v_c + v^2 u_a u_c$ where $v^2 = v^a v_a$ and g (and not g') is used to raise and lower indices. Also we can write $g'^{ab} = g^{ab} - p^{ab}$ where $p^{ab} = v^a u^b + u^a v^b + v^a v^b$. Let $\left\{ {a \atop b\ c} \right\}' = \left\{ {a \atop b\ c} \right\} - K_b{}^a{}_c$. Then we can see that

$$K_d{}^a{}_f = -\tfrac{1}{2}\left(g'_{bd}p^{ab}{}_{;f} + g'_{ef}p^{ae}{}_{;d} - g'_{ef}p^{eb}{}_{;c}g'_{bd}g'^{ca}\right)$$

and

$$\Gamma'{}_b{}^a{}_c - \Gamma_b{}^a{}_c = -K_b{}^a{}_c - v^a u_{b;c} - (u^a + v^a)u_d K_b{}^d{}_c \; .$$

(II.27) <u>Newtonian Connections</u>

A symmetric Galilei connection is said to be *Newtonian* if $\gamma^{p[a}R^{c]}{}_{(bd)p} = 0$, where $R^c{}_{bdp}$ is the Riemann tensor derived from this connection (cf. Künzle [46], p. 350). If we have a vector field u^a with $u^a u_a = -1$ and the usual associated Lorentz metric with curvature tensor $\mathcal{R}^c{}_{bdp}$, then $\gamma^{p[a}\mathcal{R}^{c]}{}_{(bd)p} = u^p u^{[a}\mathcal{R}^{c]}{}_{(bd)p}$ since $\mathcal{R}^{[c}{}_{(bd)}{}^{a]} = 0$ by symmetries of the Christoffel symbol Riemann tensor. For the basic Galilei connection (II.25.2) holds and hence $\gamma^{p[a}C^{c]}{}_{(bd)p} = u^{[a}\mathcal{R}^{c]}{}_{(bd)}{}^e u_e$. Since $R^c{}_{bdp} = \mathcal{R}^c{}_{bdp} - C^c{}_{bdp}$ we see that $\gamma^{p[a}R^{c]}{}_{(bd)p} = 0$ so that *any basic symmetric Galilei connection is a Newtonian connection.* The basic Galilei connection is equation (4.6), p. 348 of [46]. Thus if an integrable Galilei structure $(\gamma^{ab}u_b)$, $du_b\big|_c = 0$ is given on M then corresponding to any vector field u^a with $u^a u_a = -1$ everywhere, the associated basic Galilei connection is a Newtonian connection on M.

Since a Newtonian connection is a symmetric Galilei connection we see that every Newtonian connection must be of the form

$$\Gamma^{a}_{b\ c} = \left\{{a \atop b\ c}\right\} - u^a u_{b;c} - \Lambda^a_{\ b} u_c - \Lambda^a_{\ c} u_b - t^a u_b u_c \qquad \text{(II.27.1)}$$

since the contorsion is given by (II.24.1). The condition for a Newtonian connection is $\gamma^{p[a}C^{c]}{}_{(bd)p} = u^{[a}R^{c]}{}_{(bd)e}u^e$ where of course $C^a{}_{dbc} = C_{basic}{}^a{}_{dbc} + \Delta C^a{}_{dbc}$ is given by (II.25.1), and C_{basic} by (II.25.2). We have only to set $\gamma^{p[a}\Delta C^{c]}{}_{(bd)p} = 0$ since we know the required condition holds for C_{basic}. We can also write the contorsion in terms of the antisymmetric non-orthogonal $\mathring{\Lambda}_{eb}$ which incorporates both Λ_{eb} and the orthogonal vector t_b (II.24.4). In this case $K^{a}_{b\ c} = u^a u_{b;c} + 2\gamma^{ae}\mathring{\Lambda}_{e(b}u_{c)}$. In this general case we can show that

$$\begin{aligned}
2\gamma^c{}_{[p}\Delta C_{a](db)c} &= u_d\left\{d\mathring{\Lambda}_{ap}\big|_b + u_p u^c d\mathring{\Lambda}_{ba}\big|_c + u_a u^c d\mathring{\Lambda}_{pb}\big|_c\right\} \\
&\quad + u_b\left\{d\mathring{\Lambda}_{ap}\big|_d + u_p u^c d\mathring{\Lambda}_{da}\big|_c + u_a u^c d\mathring{\Lambda}_{pd}\big|_c\right\} \\
&= 2d\mathring{\Lambda}_{ef}\big|_{(b}u_{d)}\gamma^e_a\gamma^f_p \, . \qquad \text{(II.27.2)}
\end{aligned}$$

Here $d\mathring{\Lambda}_{ef}\big|_b = \mathring{\Lambda}_{ef,b} + \mathring{\Lambda}_{fb,e} + \mathring{\Lambda}_{be,f}$ is the exterior derivative of the 2-form $\mathring{\Lambda}_{ef}$ in component form. If $d\mathring{\Lambda}_{ef}\big|_b \neq 0$ then clearly $d\mathring{\Lambda}_{ef}\big|_b\gamma^e_a\gamma^f_p \neq 0$ so one can see that *a necessary and sufficient condition for* $\Gamma^{a}_{b\ c}$ *to be Newtonian is* $d\mathring{\Lambda}_{ef}\big|_b = 0$ [46, p. 350].

Now let us take a look at the difference between two basic Galilei connections on a fixed Galilei structure (γ^{ab}, u_a) corresponding to u^a and $u'^a = u^a + v^a$ where v^a is orthogonal. Recall that in (II.26) we had expressions for $K^{a}_{d\ f} = \left\{{a \atop d\ f}\right\} - \left\{{a \atop d\ f}\right\}'$ and $\Gamma'^{a}_{b\ c} - \Gamma^{a}_{b\ c}$ and we put $g'^{ab} = g^{ab} - p^{ab}$, $g'_{ab} = g_{ab} + h_{ab}$ where all these symbols were defined. We let $\alpha = \frac{v^2}{2} > 0$ for simplicity and also introduce the notation $w_f = u^b{}_{;f}v_b$ (so w_f is orthogonal) and $-w_f = u^b v_{b;f}$. Let $\dot{v}_b = u_{b;f}u^f$. Clearly $\alpha_{,f} = v^b v_{b;f}$. Letting $s_b = v_{b;f}v^f$ we can introduce scalars β, γ, δ with

$$\dot{s}_b v^b = \alpha_{,f} v^f = \beta, \qquad \dot{s}_b u^b = -w_f v^f = \gamma,$$

$$\dot{v}_b u^b = -w_f u^f = 0, \qquad \dot{v}_b v^b = \alpha_{,f} u^f = \delta.$$

Substituting into the above expression for the difference $K_{d\ f}^{\ a}$ of the two Christoffel symbols we obtain after a lengthy calculation (using g^{ab} to raise indices)

$$K_{d\ f}^{\ a} = \frac{1}{2}\left(v_f{}^{;a} - v^a{}_{;f}\right)u_d + \frac{1}{2}\left(v_d{}^{;a} - v^a{}_{;d}\right)u_f$$

$$- \frac{1}{2}(s_f u_d + u_f s_d)(u^a + v^a) - \frac{1}{2}(\dot{v}_f u_d + u_f \dot{v}_d)v^a$$

$$+ u_f u_d\left(\alpha^{,a} - \beta u^a - (\delta+\beta)v^a\right) - u_{d;f}v^a - \frac{1}{2}(v_{d;f} + v_{f;d})(u^a + v^a)$$

$$- \frac{1}{2}(w_f u_d + u_f w_d)v^a - \frac{1}{2}(\alpha_{,f}u_d + \alpha_{,d}u_f)(u^a + v^a).$$

We thus have

$$\Gamma'^{\ a}_{d\ f} - \Gamma^{\ a}_{d\ f} = -\frac{1}{2}(v_f{}^{;a} - v^a{}_{;f})u_d - \frac{1}{2}(v_d{}^{;a} - v^a{}_{;d})u_f$$

$$- u_f u_d \alpha^{,a} - \frac{1}{2}[(\dot{v}_f + w_f)u_d + (\dot{v}_d + w_d)u_f]u^a - \delta u_f u_d u^a.$$

We can then write

$$\Gamma'^{\ a}_{d\ f} - \Gamma^{\ a}_{d\ f} = -2\gamma^{ae}\mathring{\Lambda}_{e(d}u_{f)} = -\gamma^{ae}(\mathring{\Lambda}_{ed}u_f + \mathring{\Lambda}_{ef}u_d)$$

$$= -\Lambda^a{}_d u_f - \Lambda^a{}_f u_d - t^a u_f u_d.$$

We can now compare the two expressions. Multiplying by $u^f u^d$ we get $t^a = w^a + \dot{v}^a + \delta u^a + \alpha^{,a}$ which is orthogonal. Next, substituting for t^a and multiplying by u^f picks out $\Lambda^a{}_d$ and we can obtain the following form which is clearly orthogonal and antisymmetric

$$\Lambda_{ad} = \frac{1}{2}(w_d + \dot{v}_d)u_a - \frac{1}{2}(w_a + \dot{v}_a)u_d + \frac{1}{2}(v_{d;a} - v_{a;d}).$$

Finally, we obtain from $\overset{\circ}{\Lambda}_{ad} = \Lambda_{ad} + u_{[d}t_{a]}$ that $\overset{\circ}{\Lambda}_{ad} = \frac{1}{2}(v_{d;a} - v_{a;d}) + \frac{1}{2}(u_d\alpha_{,a} - u_a\alpha_{,d})$. Thus $\overset{\circ}{\Lambda}_{ad}$ is an exact 2-form which can be written alternatively as $\overset{\circ}{\Lambda}_{ad} = d\left(\frac{1}{2}v_d + \frac{1}{2}\alpha u_d\right)\Big|_a$ as an exact differential. It is clearly a closed 2-form which we could see would be necessary since every basic Galilei connection is Newtonian. Thus the connection $\Gamma'^{\,a}_{d\,f}$ which is the primed basic Galilei connection is given by

$$\begin{aligned} \Gamma'^{\,a}_{d\,f} &= \left\{ {a \atop d\;f} \right\} - u^a u_{d;f} - 2\gamma^{ae}\overset{\circ}{\Lambda}_{e(d}u_{f)}, \\ &= \left\{ {a \atop d\;f} \right\}' - u'^a u_{d;'f} = \Gamma^{a}_{d\,f} - 2\gamma^{ae}\overset{\circ}{\Lambda}_{e(d}u_{f)}. \end{aligned} \tag{II.27.3}$$

(II.28) Dynamics for the Galilei Structure

It may turn out, particularly for cosmology, that the Einstein equations may be needed to describe the structure of the universe as a whole, but locally, for the constitutive equations (pressure volume, temperature relations) usual Classical physics is adequate. For this reason, it is worth examining the simplifying approximation of the field equations in General relativity for the rest frame approximating Galilei structure for a material medium. We shall show that the equation (4.18) given in [46, p. 351] for the Ricci tensor in terms of the density is the result obtained by taking the appropriate Galilei and Newtonian approximations of Einstein's equations. It is shown [46, pp. 351-2] that these field equations yield the classical Newtonian gravitational field under an appropriate asymptotic condition. On the other hand, since this is known for the usual general relativistic field equations, the derivation we give here will also yield this result.

Let M be that portion of space-time through which a material medium moves in an irrotational geodesic motion. Let g_{ab} be the Lorentz metric, u^a the flow vector, and (γ^{ab}, u_a) the rest frame approximating Galilei structure. Let ; denote Christoffel symbol covariant differentiation and $|$ basic Galilei covariant differentiation. Of course, $u_{a;b} = \theta_{ab}$. We assume in this approximation that heat flows and internal energy storage are not significant compared to mass density, so that $T^{ab} = -\rho c^2 u^a u^b + \hat{T}^{ab}$ where $\hat{T}^{ab}$ is symmetric and orthogonal (the 3-stress tensor) and $(\rho u^a)_{;a} = 0$ (mass conservation). These assumptions are typical for local classical theory. Then $T^{ab}{}_{|c} = T^{ab}{}_{;c} - T^{db}K_d{}^a{}_c - T^{ad}K_d{}^b{}_c$ where $K_d{}^b{}_c = u^b u_{d;c}$ so $T^{ab}{}_{|b} = 0$ if $T^{ab}{}_{;b} = 0$ since $K_d{}^b{}_b = 0$ and $(T^{ab}u_b)_{;a} = 0$, i.e. $\hat{T}^{ab}u_{b;a} = 0$. Thus we have $T^{ab}{}_{|b} = 0$ and $(\rho u^a)_{|a} = 0$ and the conservation equations are written in terms of $|$. In fact if $T^{ab}{}_{;b} = F^a$ then $T^{ab}{}_{|b} = \gamma^a_d F^d$, orthogonalizing the "external force".

The basic Galilei connection is not the only one with these properties. In fact if $\emptyset$ is the covariant differentiation for an arbitrary symmetric Galilei connection (II.24.1) with $t_d = 0$ and $\Delta K_a{}^d{}_c = \Lambda^d{}_a u_c + \Lambda^d{}_c u_a$ is the contorsion difference between its contorsion and the basic contorsion then $T^{ab}{}_{\emptyset b} = T^{ab}{}_{|b} - T^{db}\Delta K_d{}^a{}_b - T^{ad}\Delta K_d{}^b{}_b$. Recall that Λ_{da} is antisymmetric and orthogonal. Therefore $\Delta K_d{}^b{}_b = 0$. Using $T^{db} = -\rho c^2 u^d u^b + \hat{T}^{db}$ where $\hat{T}^{db}$ is orthogonal, we readily see also that $T^{db}\Delta K_d{}^a{}_b = 0$. Hence $T^{ab}{}_{\emptyset b} = T^{ab}{}_{|b} = \gamma^a_b F^b$ or equals zero depending on whether or not external forces exist. We will assume $F^a = 0$ always. Since $\Delta K_b{}^a{}_a = 0$ we similarly have $(\rho u^a)_{\emptyset a} = 0$. If $t^d \neq 0$ for the symmetric Galilei contorsion, i.e. $\Delta K_a{}^d{}_c = \Lambda^d{}_a u_c + \Lambda^d{}_c u_a + t^d u_a u_c$ then we still have $\Delta K_d{}^b{}_b = 0$ however the equation

$T^{db} K_{d\ b}^{\ a} = 0$ fails in this case, and energy and momentum are not conserved. Thus, on physical grounds, we set $t_d = 0$. However the spatially nonrotating condition, as (II.24.3) shows, leaves t_d arbitrary but forces $\Lambda_{da} = 0$. The two conditions together, one kinematic, one dynamic, leave us only with the basic Galilei connection to choose out of all symmetric ones. It is the only spatially nonrotating connection satisfying $T^{ab}{}_{|b} = 0$. The Einstein equations $\mathcal{R}^{ab} - \frac{1}{2} g^{ab}\mathcal{R} = \mathcal{G}^{ab} = \kappa T^{ab}$, $\kappa = \frac{8\pi G}{c^4}$ trivially satisfy $\mathcal{G}^{ab}{}_{;b} = 0$ as a result of geometry, but also must satisfy $\mathcal{G}^{ab}{}_{|b} = 0$ for basic Galilei covariant differentiation. This implies $\frac{1}{2} u^a{}_{;a}\mathcal{R} = \mathcal{R}^{ab}u_{a;b}$ or $\frac{1}{2}\theta\mathcal{R} = \mathcal{R}^{ab}\theta_{ab}$. It is worth noting that since $u_{a,b} = u_{b,a}$ a local time τ exists on neighborhoods with $\tau_{,a} = u_a = g_{ab}u^b$.

Clearly $\mathcal{R}_{ab} = \mathcal{G}_{ab} - \frac{1}{2} g_{ab}\mathcal{G}$ where $\mathcal{G} = \mathcal{G}^a{}_a = -\mathcal{R} = -\mathcal{R}^a{}_a$, and hence $\mathcal{R}_{ab} = \kappa(T_{ab} - \frac{1}{2} g_{ab}T)$. Using $R_{ab} = \mathcal{R}_{ab} - \mathcal{C}_{ab}$ where $\mathcal{C}_{ab} = -(u^d u_{a;b})_{;d}$ (II.25) we may write (from $T_{ab} = -\rho c^2 u_a u_b + \hat{T}_{ab}$) an expression for the basic Galilei Ricci tensor as $R_{ab} = u^d{}_{;d}u_{a;b} + u^d u_{a;bd} + \kappa\left(-\frac{\rho c^2}{2} u_a u_b - \frac{\rho c^2}{2}\gamma_{ab} + \hat{T}_{ab} - \frac{1}{2}\gamma_{ab}\hat{T} + \frac{1}{2} u_a u_b \hat{T}\right)$ where $\hat{T} = \hat{T}_{ab}\gamma^{ab}$. Since $u^a u_{a;bd} = -u^a{}_{;d}u_{a;b}$ we see that $u^d u_{a;bd}$ is symmetric in a and b and orthogonal in these indices. Hence multiplication by u^a yields $R_{ab}u^a = \frac{\kappa}{2}(\rho c^2 - \hat{T})u_b = \frac{4\pi G}{c^4}(\rho c^2 - \hat{T})u_b$, so we see immediately that $R_{ab} = S_{ab} - \frac{4\pi G}{c^4}(\rho c^2 - \hat{T})u_a u_b$ where S_{ab} is orthogonal and symmetric. Orthogonalizing we find that $S_{ab} = (u^d u_{a;b})_{;d} + \kappa\hat{T}_{ab} - \frac{\kappa}{2}(\rho c^2 + \hat{T})\gamma_{ab}$.

These equations give a direct comparison between the Lorentz and Galilei cases. If $|$ is the basic Galilei derivative we have seen that $T^{ab}{}_{|b} = 0$, $T^{ab}{}_{;b} = 0$, $\hat{T}^{ab}{}_{|b} = 0$, $\hat{T}^{ab}{}_{;b} = 0$. Using g_{bc} to lower indices it is not hard to show that the equations for the Galilei case can be written as

$$(\rho u^a)_{|a} = 0, \quad \hat{T}_{ab|c}\gamma^{bc} = 0, \quad \hat{T}_{ab|c}u^b u^c = 0. \tag{II.28.1}$$

In the Newtonian case above we have made an approximation by taking the relativistic field equations which are assumed exact and seeing what they give in a case where application of Newtonian mechanics seems appropriate. Further approximations then could be called for. First of all, the principal stresses of the stress tensor $\hat{T}^{ab}$ are small compared to the rest mass density ρc^2, i.e. $\hat{T}_{ab} << \rho c^2 \gamma_{ab}$ and $\hat{T} << \rho c^2$. Hence our field equations become

$$R_{ab} = S_{ab} - \frac{4\pi G}{c^2}\rho u_a u_b \quad \text{and} \quad S_{ab} = (u^d u_{a;b})_{;d} - \frac{\kappa}{2}\rho c^2 \gamma_{ab} .$$

Now the mass density is the only source of the gravitational field and the stress does not contribute — a result we would expect for a non-relativistic gravitational theory.

In Newtonian theory any velocity vector v^a of the motion of a particle or frame of reference must be small compared to the velocity of light in order for the Galilei transformation to be appropriate. This means that v^a differs only slightly from u^a the flow field of the material medium, v^a being time like, $v^a v_a = -1$. We can then write $v^a = u^a + \epsilon h^a$ where $h^a u_a = 0$, $h^a h_a = 1$, $\epsilon > 0$, $\epsilon << 1$ to first order in ϵ. The answer as to which terms "dominate" in the above expression for R_{ab} can be answered by contracting with $v^a v'^b$ and seeing which terms are of order ϵ^2 to be disposed of $(v'^a = u^a + \epsilon' h'^a)$. Since S_{ab} is orthogonal its contribution is of order ϵ^2 so that

$$R_{ab} = -\frac{4\pi G}{c^2}\rho u_a u_b \tag{II.28.2}$$

are the Newtonian-Galileian field equations in agreement with [46, p. 351].

CHAPTER III

SOLUTION OF EINSTEIN'S EQUATIONS USING RICCI COEFFICIENTS

(III.1) Lorentz and Riemann Structures on Manifolds.

In this chapter we return to the use of frame components and "coordinate torsion" associated with a reference chart for a manifold equipped with a group structure that was discussed in (II.20). At this time it would probably be good to review that section before proceeding to read this chapter. The notation used here will be that of (II.20) and we will refer to the $T^{\alpha}{}_{j}{}^{i}{}_{k}$ as the *Ricci (Rotation) Coefficients* for the local frame component system on M with structure $G(M)$ associated with the reference chart indexed by α. Our purpose here will be to use frame components and these Ricci coefficients to find new techniques for solving the field equations of general relativity. As we proceed the utility and power of this method will become evident.

Let M be a differential manifold with a smooth Lorentz or Riemann metric g. In general non-homogeneous coordinates (i.e. frame components) the Christoffel symbol connection is determined from $g_{ij;k} = 0$ and $T_{\alpha j}{}^{i}{}_{k} = 0$, i.e. $g_{ij,k} - g_{\ell j}\left\{{}_{i}{}^{\ell}{}_{k}\right\} - g_{i\ell}\left\{{}_{j}{}^{\ell}{}_{k}\right\} = 0$ and $0 = T_{\alpha j}{}^{i}{}_{k} = -T^{\alpha}{}_{j}{}^{i}{}_{k} + \left\{{}_{j}{}^{i}{}_{k}\right\} - \left\{{}_{k}{}^{i}{}_{j}\right\}$. In these "coordinates" g_{ij} is symmetric but $\left\{{}_{j}{}^{i}{}_{k}\right\}$ is not, while $\left\{{}_{j}{}^{i}{}_{k}\right\}$ is symmetric in homogeneous coordinates (II.20) which is what we usually think of when referring to coordinates. Permuting and adding the equation involving $g_{ij,k}$ we obtain

$$g_{ij,k} + g_{jk,i} - g_{ki,j} = g_{\ell k}\left(\left\{{\ell \atop j\ i}\right\} - \left\{{\ell \atop i\ j}\right\}\right) + g_{i\ell}\left(\left\{{\ell \atop j\ k}\right\} - \left\{{\ell \atop k\ j}\right\}\right)$$

$$+ g_{j\ell}\left(\left\{{\ell \atop i\ k}\right\} + \left\{{\ell \atop k\ i}\right\}\right)$$

$$= g_{\ell k}T^{\alpha}{}_{j}{}^{\ell}{}_{i} + g_{i\ell}T^{\alpha}{}_{j}{}^{\ell}{}_{k} + g_{j\ell}\left(\left\{{\ell \atop i\ k}\right\} + \left\{{\ell \atop k\ i}\right\}\right)$$

Hence,

$$\left\{{\ell \atop i\ k}\right\} + \left\{{\ell \atop k\ i}\right\} = g^{j\ell}\left(g_{ij,k} + g_{jk,i} - g_{ki,j} - g_{ak}T^{\alpha}{}_{j}{}^{a}{}_{i} - g_{ia}T^{\alpha}{}_{j}{}^{a}{}_{k}\right),$$

but $\left\{{\ell \atop i\ k}\right\} - \left\{{\ell \atop k\ i}\right\} = T^{\alpha}{}_{i}{}^{\ell}{}_{k}$ and so therefore

$$\left\{{\ell \atop i\ k}\right\} = \frac{1}{2}\, g^{j\ell}\left(g_{ij,k} + g_{jk,i} - g_{ki,j} - g_{ak}T^{\alpha}{}_{j}{}^{a}{}_{i} - g_{ia}T^{\alpha}{}_{j}{}^{a}{}_{k}\right) + \frac{1}{2}\, T^{\alpha}{}_{i}{}^{\ell}{}_{k}$$

and the two terms divide $\left\{{\ell \atop i\ k}\right\}$ into its symmetric and antisymmetric part. Of course in homogeneous coordinates where $T^{\alpha} = 0$ this reduces to the usual formula for the Christoffel symbols.

Now we can work in a reference chart of the Lorentz or Riemann metric group structure itself (i.e. that preserves the metric structure) such that the components g_{ij} are constant and equal to $\eta_{ij} = \mathrm{diag}_{ij}(1,1,\cdots,1,-1)$ for the Lorentz case, or $\eta_{ij} = \delta_{ij}$ for the Riemann case. Here, all the derivatives of g_{ij} vanish, and we may write

$$\left\{{\ell \atop i\ k}\right\} = \frac{1}{2}\, T^{\alpha}{}_{i}{}^{\ell}{}_{k} - \frac{1}{2}\, \eta^{j\ell}\left(\eta_{ak}T^{\alpha}{}_{j}{}^{a}{}_{i} + \eta_{ia}T^{\alpha}{}_{j}{}^{a}{}_{k}\right). \qquad \text{(III.1.1)}$$

From this we can calculate the Riemann tensor components in this generalized coordinate (frame component) system using the general formula involving coordinate torsion which we obtained in (II.20). Because of the nature of the coordinates we are using, we actually appear to reduce General Relativity to Special Relativity, because the simple diagonal constant metric η_{ij} is used to raise and lower indices.

All the information about the curvature is contained in the coordinate torsion. Furthermore, Einstein's second order field equations are reduced in this case to first order equations for the coordinate torsion which are linear in the first derivative.

We can write (as in (II.20)),

$$R^{i}{}_{jk\ell} = \left\{ {i \atop j\ \ell} \right\}_{,k} - \left\{ {i \atop j\ k} \right\}_{,\ell} + \left\{ {i \atop a\ k} \right\}\left\{ {a \atop j\ \ell} \right\} - \left\{ {i \atop a\ \ell} \right\}\left\{ {a \atop j\ k} \right\} + \left\{ {i \atop j\ a} \right\} T^{\alpha}{}_{k}{}^{a}{}_{\ell} \tag{III.1.2}$$

where the comma refers (as above and in the remainder of this chapter) to the derivative with respect to the natural parameter along the integral curve of the vector field identified by the number following the comma. This vector field is the coordinate constant vector field obtained from the appropriate base vector in V, the $n+1$ dimensional vector space of the group structure $\mathcal{G}(M)$ where $\dim M = n+1$. See (III.2) below for a discussion of coordinate constant vector fields.

Because η_{ij} is constant, index lowering commutes with this generalized "partial differentiation" and so we have,

$$R_{ijk\ell} = \{ji\ell\}_{,k} - \{jik\}_{,\ell} + \{aik\}\left\{ {a \atop j\ \ell} \right\} - \{ai\ell\}\left\{ {a \atop j\ k} \right\} + \{jia\} T^{\alpha}{}_{k}{}^{a}{}_{\ell}.$$

Even though the coordinate torsion is not a tensor, we can use the η_{jk} to lower indices and obtain $\big(\{i\ell k\} + \{\ell ik\} = 0\big)$ $\{i\ell k\} = \frac{1}{2} T^{\alpha}{}_{i\ell k} - \frac{1}{2} T^{\alpha}{}_{\ell ki} - \frac{1}{2} T^{\alpha}{}_{\ell ik}$, and also

$$\left\{ {\ell \atop i\ k} \right\} = \frac{1}{2} T^{\alpha}{}_{i}{}^{\ell}{}_{k} - \frac{1}{2} T^{\alpha\ell}{}_{ki} - \frac{1}{2} T^{\alpha\ell}{}_{ik}. \tag{III.1.3}$$

There are actually an infinite number of such (generalized) coordinate systems consistent with the metric, and these correspond to the choices of bundle charts containing a particular $x \in M$ in the

tangent bundle associated with the group structure on M. The condition $g_{ij} = \eta_{ij}$ can be achieved by an appropriate choice of fixed basis for V. This basis must be a Lorentz basis with respect to the Lorentz inner product I on V which generates the metric tensor on M as an invariant tensor field in the way discussed in Chapter II. To keep our notation consistent with the first two chapters we use $\mathcal{R}$ instead of R when talking about the Christoffel symbol Riemann tensor, this being the only one of importance here. In general we restrict ourselves to (generalized) coordinate systems compatible with the group structure, and if the space is not locally homogeneous, we may not be able to find torsion free coordinates (of this type) at every point.

(III.2) Coordinate Constant Vector Fields.

Let (U_α, ϕ_α) be a bundle chart $[(U_\alpha, r_\alpha)$ the corresponding reference chart (II.1)] for a group structure $\mathcal{G}(M)$. A vector field on U_α is said to be a *coordinate constant field* with respect to α if it is the lift through ϕ_α of a fixed vector in V. Hence, in component form, if X^i is such a field, then $X^i{}_{,j} = 0$ for all $i,j = 1,2,\cdots,n+1 = \dim M$. If X^i, Y^j are coordinate constant on U_α then (II.20), $[X,Y]^k = T^{\alpha}{}_{i}{}^{k}{}_{j} Y^i X^j$ which is not in general coordinate constant, i.e. Y^i and X^j as components are constant for each $x \in U_\alpha$ but $T^{\alpha}{}_{i}{}^{k}{}_{j}$ is a function of x.

(III.3) Jacobi's Identity for the Coordinate Torsion.

An important and powerful relation involving the coordinate torsion that holds in general can be obtained by using the formula $[X,Y]^k = Y^k{}_{,\ell} X^\ell - X^k{}_{,\ell} Y^\ell + T^{\alpha}{}_{j}{}^{k}{}_{\ell} Y^j X^\ell = W^k$ from (II.20) in the Jacobi

Identity $[[X,Y],Z]+[[Y,Z],X]+[[Z,X],Y] = 0$. We put $[[X,Y],Z]^c =$ $[W,Z]^c = Z^c{}_{,d}W^d - W^c{}_{,d}Z^d + T^{\alpha}{}_a{}^c{}_b Z^a W^b$ and substitute for W^k above. Then cyclically permuting X, Y, Z, adding and simplifying, using the fact that the equation must be true for all fields X^a, Y^a, Z^a we obtain

$$T^{\alpha}{}_j{}^c{}_{i,k} + T^{\alpha}{}_k{}^c{}_{j,i} + T^{\alpha}{}_i{}^c{}_{k,j} = T^{\alpha}{}_k{}^c{}_b T^{\alpha}{}_j{}^b{}_i + T^{\alpha}{}_i{}^c{}_b T^{\alpha}{}_k{}^b{}_j + T^{\alpha}{}_j{}^c{}_b T^{\alpha}{}_i{}^b{}_k \,.$$

This important result is strictly a consequence of the associative property of the composition (of point derivative operators which vector fields are) in the relation $[X,Y] = XY - YX$. The derivation can be simplified somewhat by the use of coordinate constant vector fields if desired. This result is valid in general frame components — metric or not. We may alternatively write $T^{\alpha}{}_{[j}{}^c{}_{i,k]} + T^{\alpha}{}_b{}^c{}_{[k} T^{\alpha}{}_j{}^b{}_{i]} = 0$ as the Jacobi identity for the coordinate torsion.

(III.4) The Riemann Tensor and Einstein's Equations.

Using the Jacobi identity we find that the Riemann Tensor for the Christoffel symbols can be expressed in metric frame components as

$$\begin{aligned}
R_{ijk\ell} &= \tfrac{1}{2}\left(T_{ki\ell,j} - T_{kj\ell,i} + T_{ikj,\ell} - T_{i\ell j,k}\right) \\
&\quad - \tfrac{1}{2} C_{ijk\ell} + \tfrac{1}{4}\left(C_{i\ell jk} - C_{ikj\ell}\right) + \tfrac{1}{4}\left(F_{i\ell kj} + F_{i\ell jk} + F_{\ell ikj} + F_{\ell ijk}\right) \\
&\quad - \tfrac{1}{4}\left(F_{ik\ell j} + F_{ikj\ell} + F_{ki\ell j} + F_{kij\ell}\right) + \tfrac{1}{4}\left(M_{kij\ell} - M_{ikj\ell} + M_{\ell jik} - M_{j\ell ik}\right) \\
&\quad - \tfrac{1}{4}\left(M_{\ell ijk} - M_{i\ell jk} + M_{kji\ell} - M_{jki\ell}\right),
\end{aligned} \qquad \text{(III.4.1)}$$

where

$$M_{ikj\ell} = T_{aik}\, T_j{}^a{}_\ell \qquad (M_{ikj\ell} + M_{ik\ell j} = 0),$$

$$F_{ik\ell j} = T_{aik}\, T^a{}_{\ell j} \qquad (F_{ik\ell j} = F_{\ell jik}),$$

$$C_{ikj\ell} = T_{iak}\, T_j{}^a{}_\ell \qquad \begin{pmatrix} C_{ikj\ell} = -C_{kij\ell} \\ = C_{j\ell ik} = -C_{ik\ell j} \end{pmatrix},$$

and where α has been dropped from T as we understand this to represent the coordinate torsion. Indices on T, which is not a tensor, have been freely raised and lower using η_{ij} and its inverse η^{ij}. As there is no connection being considered with non-zero torsion, the α index on the coordinate torsion can be dropped without confusion.

We have not as yet entirely specified the frame component system to the degree we would like simply by requiring it to be metric. We can view the metric frame component system as a smooth association of an orthonormal tetrad at each point of $U_\alpha \subset M$ with a fixed orthonormal tetrad in V. Consequently we can further specify our frame component system by requiring the symmetric Ricci tensor to be in the simplest possible canonical form. In the case of a Reimannian structure, where the metric is positive definite, and $\eta_{ij} = \delta_{ij}$ we can choose the metric frame components so as to diagonalize the Ricci tensor R_{jk}. In the Lorentz case where $\eta_{ij} = 0$, $i \neq j$, $\eta_{ii} = 1$, $1 \leq i \leq n$, $\eta_{n+1,n+1} = -1$ we can write the stress tensor $\hat{T}^{ab}$ and the energy momentum tensor T^{ab} in the case $n = 3$ as

$$T^{ab} = -(\rho c^2 + \epsilon)u^a u^b - \lambda u^a v^b - \lambda v^a u^b + \hat{T}^{ab} \qquad \text{(III.4.2)}$$

where u^a is time-like, $u^a u_a = -1$, $u_a \hat{T}^{ab} = 0$, $u_a v^a = 0$, $v_a v^a = 1$, and $\hat{T}^{ab} = \sigma_1 r^a r^b + \sigma_2 s^a s^b + \sigma_3 t^a t^b$ where (r^a, s^a, t^a) is an orthonormal triad orthogonal to u^a and σ_i are the principal stresses. Through the Einstein field equations $R_{ij} - \frac{1}{2}\eta_{ij}R = \kappa T_{ij}$ where $\kappa = 8\pi G/c^4$ we see that the Ricci tensor is in a simple reduced form. Let $v^a = \alpha^1 r^a + \alpha^2 s^a + \alpha^3 t^a$ where $(\alpha_1)^2 + (\alpha_2)^2 + (\alpha_3)^2 = 1$. In adapted (generalized) coordinates $u^a = -u_a = \delta_{a4}$, $r^a = r_a = \delta_{a1}$, $s^a = s_a = \delta_{a2}$, $t^a = t_a = \delta_{a3}$ for $a = 1,2,3,4$. We can then write

$$T^{ab} = -(\rho c^2+\epsilon)u^a u^b - \lambda u^a(\alpha^1 r^b+\alpha^2 s^b+\alpha^3 t^b) - \lambda u^b(\alpha^1 r^a+\alpha^2 s^a+\alpha^3 t^a)$$
$$+ \sigma_1 r^a r^b + \sigma_2 s^a s^b + \sigma_3 t^a t^b$$

in matrix form in adapted frame components as

$$(T^{ab}) = \begin{pmatrix} \sigma_1 & 0 & 0 & -\lambda\alpha^1 \\ 0 & \sigma_2 & 0 & -\lambda\alpha^2 \\ 0 & 0 & \sigma_3 & -\lambda\alpha^3 \\ -\lambda\alpha^1 & -\lambda\alpha^2 & -\lambda\alpha^3 & -(\rho c^2+\epsilon) \end{pmatrix} \tag{III.4.3}$$

and T_{ab} has the same components except the minus sign is dropped from before each λ term. The α^i terms are dimensionless while λ and $\rho c^2 + \epsilon$ have dimensions of pressure or energy density. By contracting the Einstein equations in adapted coordinates with the metric η^{ij} we can rewrite them as $R_{ij} = \kappa\left(T_{ij} - \frac{1}{2} T\eta_{ij}\right)$ where $T = T_{ij}\eta^{ij}$, and from this we have the matrix of Ricci tensor components being given by

$$R_{11} = -\frac{\kappa}{2}\left(\rho c^2+\epsilon-\sigma_1+\sigma_2+\sigma_3\right), \qquad R_{22} = -\frac{\kappa}{2}\left(\rho c^2+\epsilon+\sigma_1-\sigma_2+\sigma_3\right)$$

$$R_{33} = -\frac{\kappa}{2}\left(\rho c^2+\epsilon+\sigma_1+\sigma_2-\sigma_3\right), \qquad R_{44} = -\frac{\kappa}{2}\left(\rho c^2+\epsilon-\sigma_1-\sigma_2-\sigma_3\right)$$

$$R_{12} = R_{13} = R_{23} = R_{21} = R_{31} = R_{32} = 0, \quad \text{and}$$

$$R_{i4} = R_{4i} = \kappa\lambda\alpha^i, \quad i = 1,2,3. \tag{III.4.4}$$

We can obtain the Ricci tensor R_{jk} in adapted coordinates in terms of the coordinate torsion and its first derivative by contracting the long expression (III.4.1) we had for $R_{ijk\ell}$ with $\eta^{i\ell}$ which is a constant and has zero "partial" as well as zero covariant derivative in these frame components. We find that

$$R_{jk} = T_{(k,j)} + T^i{}_{(kj),i} + \frac{1}{2} B_{jk} - \frac{1}{4} E_{jk} + A_{(jk)} + \frac{1}{2} G_{jk} \tag{III.4.5}$$

where $T_k = T_{k\ i}^{\ i}$, $B_{jk} = T^i_{\ aj}T_{i\ k}^{\ a} = B_{kj}$, $A_{kj} = T_a T^a_{\ kj}$, $E_{kj} = T_{ak}^{\ \ i}T^a_{\ ji} = E_{jk}$ and $G_{kj} = T_{a\ k}^{\ i}T_{i\ j}^{\ a} = G_{jk}$. Similarly we can obtain the curvature scalar $R = \eta^{jk}R_{jk}$ as $R = 2T^k_{\ ,k} + T_a T^a + \frac{1}{4} B^k_k + \frac{1}{2} G^k_k$, where $B^k_k = E^k_k = T_{aki}T^{aki}$, $A^k_k = T^a T_a$ and $G^k_k = T_a^{\ ik}T_{i\ k}^{\ a}$.

Provided we are dealing with a group structure that admits transformations with positive determinant only (which has been assumed), the completely skew symbol ε_{abcd} transforms like a tensor. In adapted coordinates $\varepsilon_{abcd} = -\epsilon_{abcd}$ where $\epsilon_{1234} = 1$ and we are considering $n + 1 = 4$ here. Raising indices using η^{ab} we have $\varepsilon^{abcd} = \epsilon^{abcd}$ where $\epsilon^{abcd} = \epsilon_{abcd}$ (I.11). Furthermore since $\left\{ {a \atop a\ e} \right\} = 0$ in adapted coordinates, and $\epsilon_{abcd,e} = 0$ we have $\varepsilon_{abcd;e} = 0$ as we have seen already. Likewise $\varepsilon^{abcd}_{\ \ \ \ ,e} = 0 = \varepsilon^{abcd}_{\ \ \ \ ;e}$.

Now $u_{a;b} = u_{a,b} - u_c\left\{ {c \atop a\ b} \right\} = -u_4\left\{ {4 \atop a\ b} \right\} = \left\{ {4 \atop a\ b} \right\}$ in adapted (generalized) coordinates. Hence from (III.1.3), $u_{a;b} = \frac{1}{2}\left(T_{a\ b}^{\ 4} - T^4_{\ ba} - T^4_{\ ab}\right)$ so that clearly $u_{4;b} = 0$, i.e. $u^a u_{a;b} = 0$. Likewise $\dot{u}_a = u_{a;b}u^b$ is given by $\dot{u}_a = T_{a\ 4}^{\ 4}$ and $\dot{u}_4 = 0$.

As we let a, b, c, etc. run from 1 up to 4 we let A, B, C, D run from 1 to 3 to cover the orthogonal part. Define $\varepsilon_{abc} = \varepsilon_{abcd}u^d$, $\varepsilon^{abc} = \varepsilon^{abcd}u_d$, $u_{a:b} = u_{a;b} + \dot{u}_a u_b$. Then ε_{abc}, ε^{abc}, $u_{a:b}$ are orthogonal, i.e. any contraction with u^a or u_a on any index vanishes. In adapted coordinates we may write $\varepsilon_{ABC} = -\epsilon_{ABC}$, $\varepsilon^{ABC} = -\epsilon^{ABC}$ where $\epsilon_{123} = \epsilon^{123} = 1$ and ϵ is completely skew in 3 space, and we understand that any index involving a 4 means a zero value for the tensor. Then we have $u_{A:B} = \frac{1}{2}\left(T_{A\ B}^{\ 4} - T^4_{\ BA} - T^4_{\ AB}\right)$. Setting $\omega_{AB} = u_{[A:B]}$ and $\theta_{AB} = u_{(A:B)}$ we have that

$$\omega_{AB} = \frac{1}{2}\, T_{A\ B}^{\ 4} \quad \text{and} \quad \theta_{AB} = -\frac{1}{2}\left(T^4_{\ BA} + T^4_{\ AB}\right), \quad A,B = 1,2,3.$$

If we let $\theta = \delta^{AB}\theta_{AB}$ then $\theta = -T^4 = T_4$ where $T_k = \eta^{i\ell}T_{ki\ell}$. Thus $\theta = T_{4\ i}^{\ i} = T_{4\ I}^{\ I}$ determines the expansion rate of the material medium along flow lines (I.11). If we let the vorticity (I.16) be given by $\upsilon^A = \frac{1}{2}\varepsilon^{ABC}\omega_{BC}$ where conversely $\omega_{AB} = \varepsilon_{ABC}\upsilon^C$ we have $\upsilon^1 = -\frac{1}{2}T_{2\ 3}^{\ 4} = -\omega_{23}$, $\upsilon^2 = -\frac{1}{2}T_{3\ 1}^{\ 4} = -\omega_{31}$ and $\upsilon^3 = -\frac{1}{2}T_{1\ 2}^{\ 4} = -\omega_{12}$ in adapted frame components.

(III.5) Dynamics of Motion in a Space-time.

Recall that the Energy-Momentum tensor T^{ab} was given in (III.4.2) by $T^{ab} = -(\rho c^2+\epsilon)u^a u^b - \lambda u^a v^b - \lambda v^a u^b + \hat{T}^{ab}$. Here $\hat{T}^{ab}$ is the symmetric orthogonal stress tensor and $\rho c^2 > 0$, $\epsilon > 0$, $(\rho c^2 u^a)_{;a} = 0$, $v^a v_a = 1$, $v^a u_a = 0$. We interpret ρ as the mass density, ϵ as the thermal energy density, λv^a as the heat flow vector, λ the magnitude and v^a the orthogonal normalized direction. If $\tilde{u}^a$ is another time-like unit vector representing a speed of motion $v = c \tanh \gamma$ in a space direction determined by $h^a (h^a h_a = 1,\ h^a u_a = 0)$ relative to the rest frame of the material medium which has four-velocity u^a (the flow vector) we have $\tilde{u}^a = (\cosh \gamma)u^a + (\sinh \gamma)h^a$, $\gamma \in \mathbb{R}$. If we let $\hat{T}^{ab} = \sigma_1 r^a r^b + \sigma_2 s^a s^b + \sigma_3 t^a t^b$ where u^a, r^a, s^a, t^a form an orthonormal tetrad, and if $\sigma_{max} = \max\{|\sigma_1|, |\sigma_2|, |\sigma_3|\}$ then $0 > T^{ab}\tilde{u}_a\tilde{u}_b$ for all γ and h^a if $\rho c^2 + \epsilon \geq 2\lambda + \sigma_{max}$. This is the basic condition of positive energy density in any frame that we impose on T^{ab}. It is to be noted that the condition that $T^{ab}\tilde{u}_a\tilde{u}_b$ be *negative* is simply a consequence of our choice $\tilde{u}_a\tilde{u}^a = -1$ instead of $+1$. In this case $T^a_{\ b}$ maps future pointing time-like vectors to future pointing time-like vectors, indicating *positive* energy density, if $T^a_{\ b}\tilde{u}^b$ is time-like. In fact, $T^{ab}\tilde{u}_b$ is not space-like if $\rho c^2 + \epsilon \geq 2\lambda + \sigma_{max}$.

Putting $T^{ab}{}_{;b} = 0$ and contracting with u_a we obtain

$$\underbrace{\hat{T}^{ab}\theta_{ab} - \lambda v^a \dot{u}_a}_{\substack{\text{Energy flow into system per}\\ \text{unit 4-volume due to work}\\ \text{done on the system by}\\ \text{stresses and heat flux.}}} = \underbrace{(\epsilon u^b)_{;b}}_{\substack{\text{Internal energy gained}\\ \text{by the system per unit}\\ \text{4-volume.}}} + \underbrace{(\lambda v^b)_{;b}}_{\substack{\text{Heat flow out of}\\ \text{the system to the}\\ \text{environment per}\\ \text{unit 4-volume.}}} .$$

We refer to this as the *equation of conservation of energy in the rest frame.* If we let $\gamma^a_c = \delta^a_c + u^a u_c$, $\gamma_{ac} = g_{ac} + u_a u_c$ and take the spacelike part of $T^{ab}{}_{;b} = 0$ we get

$$\underbrace{(\rho c^2 + \epsilon)\dot{u}^a}_{\substack{\text{mass x acceleration}\\ \text{of body per unit}\\ \text{3-volume.}}} = \underbrace{\hat{T}^{cb}{}_{;b}\gamma^a_c}_{\substack{\text{3-force on body}\\ \text{per unit 3-volume}\\ \text{due to contact}\\ \text{stresses}}} - \underbrace{\left(\lambda u^a{}_{;b} v^b + \lambda_{,b} u^b v^a + \lambda\theta v^a + \lambda v^c{}_{;b} u^b \gamma^a_c\right)}_{\substack{\text{Momentum transfer from system}\\ \text{to environment due to heat flow}\\ \text{(per unit 3-volume per unit time).}}}$$

as the *momentum conservation equation* in the rest frame.

Of course the condition $T^{ab}{}_{;b} = 0$ giving conservation of energy and momentum is automatically implied by the Einstein field equations using the contracted Bianchi identity. In fact if we use the equation $B_{a|bc} - B_{a|cb} = B_d R^d{}_{abc} - B_{a|e} T_b{}^e{}_c$ from (II.25.3) where $\Gamma_b{}^a{}_c$ is a connection on a group structure and $T_b{}^e{}_c$ are the components of the torsion tensor, we can antisymmetrize in a, b, c and simplify to obtain

$$R^d{}_{[abc]} = -T_{[a}{}^d{}_{b|c]} + T_e{}^d{}_{[a}T_b{}^e{}_{c]} \qquad \text{(III.5.1)}$$

which is the basic *cyclic identity.* In a similar way by evaluating $B_{a|bcd} - B_{a|bdc}$ and $2B_{a|[bc]|d}$ using (II.25.3), antisymmetrizing in bcd and comparing we obtain (using the above cyclic identity)

$$R^e{}_{a[bc|d]} = R^e{}_{af[b}T_c{}^f{}_{d]} , \qquad \text{(III.5.2)}$$

which is the general *Bianchi identity*. In particular on a Lorentz space with Christoffel symbol connection $\left\{ {a \atop b\ c} \right\}$ the torsion tensor is zero, so $R^{d}{}_{[abc]} = 0$ and $R^{e}{}_{a[bc;d]} = 0$ (see (II.25.5)).

We can evaluate the energy and momentum flow terms in adapted coordinates. For example $\theta_{ab}\hat{T}^{ab} = -(T^{4}{}_{11}\sigma_1 + T^{4}{}_{22}\sigma_2 + T^{4}{}_{33}\sigma_3) = T_{411}\sigma_1 + T_{422}\sigma_2 + T_{433}\sigma_3 = T_4{}^{1}{}_{1}\sigma_1 + T_4{}^{2}{}_{2}\sigma_2 + T_4{}^{3}{}_{3}\sigma_3$. Using the summation convention (explained below) $\theta_{ab}\hat{T}^{ab} = T_4{}^{I}{}_{I}\sigma_I$, $(\epsilon u^b)_{;b} = \epsilon T_4{}^{I}{}_{I} + \epsilon_{,4}$, $(\lambda v^b)_{;b} = (\lambda\alpha^I)_{,I} + \lambda\alpha^I T_I{}^{b}{}_{b}$, $\lambda v^a \dot{u}_a = \lambda\alpha^I T_I{}^{4}{}_{4}$. Thus the energy equation in adapted coordinates is

$$\underbrace{T_4{}^{I}{}_{I}\sigma_I - \lambda\alpha^I T_I{}^{4}{}_{4}}_{\text{Work done on system}} = \underbrace{\epsilon T_4{}^{I}{}_{I} + \epsilon_{,4}}_{\text{Internal energy gained}} + \underbrace{(\lambda\alpha^I)_{,I} + \lambda\alpha^I T_I{}^{b}{}_{b}}_{\text{Heat lost to environment}} .$$

Of course we also have $\rho T_4{}^{I}{}_{I} + \rho_{,4} = 0$ since $(\rho c^2 u^a)_{;a} = 0$. Also we may write $(\rho c^2+\epsilon)\dot{u}_A = (\rho c^2+\epsilon)T_A{}^{4}{}_{4}$, and using the summation convention on repeated indices in a term which do not match those in other terms or on the other side of the equation, $\hat{T}^{cb}{}_{;b}\gamma_{cA} = \sigma_{A,A} + \sigma_A T_A{}^{b}{}_{b} - \sigma_I T_A{}^{I}{}_{I}$. (This holds for all $A = 1,2,3$ with sums on other indices.) If the equation contains any term in which the index does not appear at least twice then there is no sum - otherwise sum over it, if it is a free index not in every term. Then $\lambda u_{A;b}v^b = \lambda u_{A;B}v^B = \frac{\lambda}{2}\left(T_A{}^{4}{}_{B} + T_4{}^{B}{}_{A} + T_4{}^{A}{}_{B}\right)\alpha^B$, $\lambda_{,b}u^b v_A = \lambda_{,4}\alpha^A$, $\lambda\theta v_A = \lambda T_4{}^{i}{}_{i}\alpha^A$, $\lambda v^c{}_{;b}u^b\gamma_{Ac} = \lambda\alpha^A{}_{,4} + \lambda\alpha^I\left\{ {A \atop I\ 4} \right\} = \lambda\alpha^A{}_{,4} + \frac{\lambda}{2}\left(T_I{}^{A}{}_{4} + T_A{}^{4}{}_{I} - T_A{}^{I}{}_{4}\right)\alpha^I$. From this we obtain $(A = 1,2,3)$ the momentum equation in the form

$$\underbrace{(\rho c^2+\epsilon)T_A{}^{4}{}_{4}}_{\text{mass} \times \text{acceleration}} = \underbrace{\sigma_{A,A} + \sigma_A T_A{}^{b}{}_{b} - \sigma_I T_A{}^{I}{}_{I}}_{\text{3-force due to stresses}} - \underbrace{[\lambda(T_A{}^{4}{}_{B} + T_4{}^{B}{}_{A})\alpha^B + (\lambda_{,4} + \lambda T_4{}^{i}{}_{i})\alpha^A + \lambda\alpha^A{}_{,4}]}_{\text{Rate of transfer of momentum due to heat flow.}} .$$

(III.6) Exact and Inexact Differentials - The Exterior Derivative.

In homogeneous coordinates the exterior derivative in component form has the following form c.f. (I.28) $df|_a = f_{,a}$, $df_a|_b = f_{a,b} - f_{b,a} = 2f_{[a,b]}$, $df_{ab}|_c = f_{ab,c} + f_{bc,a} + f_{ca,b} = 3f_{[ab,c]}$ where $f_{ab} = -f_{ba}$ and similarly for higher orders, e.g. $df_{abc}|_d = f_{abc,d} - f_{bcd,a} + f_{cda,b} - f_{dab,c} = 4f_{[abc,d]}$ where $f_{abc} = f_{[abc]}$. Here d takes a k-form to a $k+1$ form, where a k-form is a completely skew covariant tensor field of rank k.

If γ represents a homogeneous system and α a frame component system with coordinate torsion $T^{\alpha}{}_{a}{}^{d}{}_{c}$ we may write $f_{\alpha a} = f_{\gamma b} G_{\gamma\alpha}{}^{b}{}_{a}$ and so (c.f. (II.20))

$$
\begin{aligned}
f_{\alpha a,c} &= f_{\gamma b\alpha,c} G_{\gamma\alpha}{}^{b}{}_{a} + f_{\gamma b} G_{\gamma\alpha}{}^{b}{}_{a,c} \\
&= f_{\gamma b,d} G_{\gamma\alpha}{}^{d}{}_{c} G_{\gamma\alpha}{}^{b}{}_{a} + f_{\gamma b} G_{\gamma\alpha}{}^{b}{}_{a,c} \quad \text{and}
\end{aligned}
$$

$$
f_{\alpha c,a} = f_{\gamma b,d} G_{\gamma\alpha}{}^{d}{}_{a} G_{\gamma\alpha}{}^{b}{}_{c} + f_{\gamma b} G_{\gamma\alpha}{}^{b}{}_{c,a} \quad \text{so}
$$

$$
\begin{aligned}
f_{\alpha a,c} - f_{\alpha c,a} &= (f_{\gamma b,d} - f_{\gamma d,b}) G_{\gamma\alpha}{}^{d}{}_{c} G_{\gamma\alpha}{}^{b}{}_{a} + f_{\gamma b}(G_{\gamma\alpha}{}^{b}{}_{a,c} - G_{\gamma\alpha}{}^{b}{}_{c,a}) \\
&= df_{\gamma b}|_d G_{\gamma\alpha}{}^{d}{}_{c} G_{\gamma\alpha}{}^{b}{}_{a} + f_{\alpha d} G_{\alpha\gamma}{}^{d}{}_{b}(G_{\gamma\alpha}{}^{b}{}_{a,c} - G_{\gamma\alpha}{}^{b}{}_{c,a}) \\
&= df_{\alpha a}|_c + f_{\alpha d} T^{\alpha}{}_{a}{}^{d}{}_{c} \, .
\end{aligned}
$$

Hence, dropping the obvious reference to α, we have

$$
df_a|_c = f_{a,c} - f_{c,a} - f_d T_a{}^d{}_c = 2f_{[a,c]} - f_d T_a{}^d{}_c \qquad \text{(III.6.1)}
$$

as the formula for exterior derivative of a 1-form in frame components. Similarly we can show that if $f_{ab} + f_{ba} = 0$,

$$
df_{ab}|_e = f_{ab,e} + f_{be,a} + f_{ea,b} + f_{cb} T_e{}^c{}_a + f_{da} T_b{}^d{}_e + f_{ce} T_a{}^c{}_b \, , \quad \text{so}
$$

$$
df_{ab}|_c = 3f_{[ab,c]} + 3f_{d[a} T_b{}^d{}_{c]} . \qquad \text{(III.6.2)}
$$

This gives us a basic idea of the form of the exterior derivative in nonhomogeneous coordinates with torsion.

Now we are in a position to consider exact and inexact differentials. If λ is a scalar and "," is frame component differentiation we have seen (II.20) that $\lambda_{,ab} - \lambda_{,ba} = T_{a\ b}^{\ c}\lambda_{,c}$. If we take $\lambda = x^i$ the i-th component of a chart function in a homogeneous coordinate system, then $x^i_{,c} = (\lambda_{,c})^i = \frac{dx^i}{\partial^c} = v^i_{(c)}$ is the i-th homogeneous coordinate of $v_{(c)}$ the c-th vector field of $(v_{(1)},\cdots,v_{(n)})$ that determine the frame component system (if $\dim M = n$). Hence

$$v^i_{(b),a} - v^i_{(a),b} = T_{b\ a}^{\ c}v^i_{(c)}. \qquad \text{(III.6.3)}$$

If $\natural$ is the partial derivative in the homogeneous coordinate system $x^1,\cdots,x^n$ then $\lambda_{,d} = \lambda_{\natural i}v^i_{(d)}$ for any scalar field λ. In particular $x^j_{,d} = x^j_{\natural i}v^i_{(d)} = \delta^j_i v^i_{(d)} = v^j_{(d)}$ as we have seen. The basis $(v_{(1)},\cdots,v_{(n)})$ can also be denoted by $\left(\frac{d}{\partial^1},\cdots,\frac{d}{\partial^n}\right)$ and the dual basis of 1-forms by $(\partial^1,\cdots,\partial^n)$. We can, of course, evaluate these at particular points $x \in U_\alpha \subset M$ say $\left.\frac{d}{\partial^a}\right|_x \in M_x$ and $\partial^b\big|_x \in M^*_x$. Likewise the homogeneous coordinate system $(x^1,\cdots,x^n)$ has a basis at x of M_x for each $x \in U_\gamma$, namely $\left(\frac{\partial}{\partial x^1},\cdots,\frac{\partial}{\partial x^n}\right)$ and a dual basis of M^*_x at x namely $(dx^1,\cdots,dx^n)$. We have that $\left[\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right] = 0$ and $d(dx^i) = 0$. The corresponding results do not hold for inexact derivatives and differentials. We have $\left[\frac{d}{\partial^a},\frac{d}{\partial^b}\right] = [\underset{\sim}{v}_{(a)},\underset{\sim}{v}_{(b)}] = T_{b\ a}^{\ c}\underset{\sim}{v}_{(c)} = T_{b\ a}^{\ c}\frac{d}{\partial^c}$, and $d(\partial^{(a)})$ in components is $d(\partial^{(a)}_b)\big|_c = T_{c\ b}^{\ a}$ in frame components, since $\partial^{(a)}_b = \delta^a_b$ (the dual basis) using (III.6.1). From (III.6.2) the condition $d\left(d(\partial^{(a)})\right) = 0$ is seen to be equivalent to the Jacobi identity

$T_{[b\ c,d]}^{\ a} + T_{e\ [b}^{\ a}T_{c\ d]}^{\ e} = 0$. Since a closed 2-form is locally exact, this gives us an existence condition on the dual basis $\partial^{(a)}$.

Of course $v^j_{(d)} = G_{\gamma\alpha\ d}^{\ \ j}$ is the transformation (II.20) between the homogeneous coordinate system γ and the frame component system α. We may write $T_{b\ a}^{\ c}v^i_{(c)} = T_{b\ a}^{\ c}G^i_{\ c} = G^i_{\ b,a} - G^i_{\ a,b}$ if we use indices i, j, k etc. for the γ system and a, b, c, etc. for the α system and it is understood that $G^i_{\ c} = G_{\gamma\alpha\ c}^{\ \ i}$. The Jacobi identity can be written as $v^i_{[(d)}T_{b\ c]\varsigma i}^{\ a} + T_{e[b}^{\ a}T_{c\ d]}^{\ e} = 0$ and for $n = 4$ if $T_{e\ b}^{\ a} = -T_{b\ e}^{\ a}$ are smooth fields defined (as scalars) on some open set in U_γ then we have a set of 16 linear equations in 16 unknowns $(v^i_{(d)})$ to solve for the vector fields. For $n = 4$ we have $(T_{b\ c}^{\ a}\epsilon^{bcdf})_{\varsigma i}v^i_{(d)} + \epsilon^{bcdf}T_{e\ b}^{\ a}T_{c\ d}^{\ e} = 0$ as a restatement of these 16 equations where $\epsilon^{1234} = 1$. It may or may not be possible to determine $v^i_{(d)}$ from these equations. Of course, if both the fields $T_{b\ c,d}^{\ a}$ and $T_{b\ c}^{\ a}$ are known, they must satisfy $T_{b\ c,d}^{\ a} = T_{b\ c\varsigma i}^{\ a}v^i_{(d)}$ for some invertible transformation $v^i_{(d)}$, and this may provide a simple way to determine $v^i_{(d)}$.

(III.7) The Cyclic and Bianchi Identities.

Our expression (III.4.1) for $R_{ijk\ell}$ in terms of $T_{j\ k}^{\ i}$ and its derivatives clearly satisfies the symmetry conditions $R_{ijk\ell} = -R_{jik\ell} = -R_{ij\ell k} = R_{k\ell ij}$. On the other hand, the condition $R_{i[jk\ell]} = 0$ when simplified, using the Jacobi identity, gives us

$$0 = T^i_{\ [kj,\ell]} - \frac{1}{2}\,T_{[kj\ell],m}\eta^{mi} + \frac{1}{2}\,T_{a[k}^{\ \ i}T_{j\ \ell]}^{\ a} + T_{a[\ell j}T^{ia}_{\ \ k]}$$

or

$$0 = T^i_{\ [kj,\ell]} - \frac{1}{2}\,T_{[kj\ell],m}\eta^{mi} + \frac{1}{2}\,M_{[k\ j\ell]}^{\ \ i} + M_{[\ell j\ k]}^{\ \ \ i}\ .$$

We call this result the *cyclic identity* for the coordinate torsion (or Ricci rotation coefficients) $T_{j\ k}^{\ i}$.

Likewise, if we write $R_{ijk\ell}$ in the form mentioned in (III.1),

$$R_{ijk\ell} = \{ji\ell\}_{,k} - \{jik\}_{,\ell} + \{aik\}\left\{{a \atop j\ \ell}\right\} - \{ai\ell\}\left\{{a \atop j\ k}\right\} + \{jia\}T_{k\ \ell}^{\ a}$$

where, of course, (III.1.3) holds, i.e. $\left\{{\ell \atop i\ k}\right\} = \frac{1}{2}\left(T_{i\ k}^{\ \ell} - T^{\ell}_{\ ik} - T^{\ell}_{\ ki}\right)$ and $\{i\ell k\} = \frac{1}{2}\left(T_{i\ell k} - T_{\ell ki} - T_{\ell ik}\right)$ so $\{i\ell k\} + \{\ell ik\} = 0$ we can evaluate $R_{ijk\ell;m} = R_{ijk\ell,m} - R_{bjk\ell}\left\{{b \atop i\ m}\right\} - R_{ibk\ell}\left\{{b \atop j\ m}\right\} - R_{ijb\ell}\left\{{b \atop k\ m}\right\} - R_{ijkb}\left\{{b \atop \ell\ m}\right\}$ by direct substitution of $R_{ijk\ell}$ above. Then antisymmetrizing to get $R_{ij[k\ell;m]}$ and equating this to zero we find the result is a trivial consequence of the Jacobi identity (III.3) and the *integrability condition* for $T_{j\ k}^{\ i}$. This is merely the statement that these coefficients of expression of the Lie brackets of the component base vector fields in terms of themselves are simply scalars, i.e. $T_{j\ k,\ell m}^{\ i} - T_{j\ k,m\ell}^{\ i} = T_{\ell\ m}^{\ c}T_{j\ k,c}^{\ i}$. Clearly by (III.1.3) the integrability condition for $T_{j\ k}^{\ i}$ is equivalent to the integrability condition for $\left\{{i \atop j\ k}\right\}$, namely $\left\{{i \atop j\ k}\right\}_{,\ell m} - \left\{{i \atop j\ k}\right\}_{,m\ell} = T_{\ell\ m}^{\ c}\left\{{i \atop j\ k}\right\}_{,c}$. Thus the Bianchi identity is a trivial consequence of the Jacobi identity and integrability conditions. A similar result holds for the cyclic identity, namely it is a consequence of the Jacobi identity and the antisymmetry condition $T_{j\ k}^{\ i} = -T_{k\ j}^{\ i}$. This can be seen by introducing a collection of scalar fields $S_{i\ jk}^{\ c}$ called the *Geometry Structure Scalars,* defined by the differential equation, $T_{i\ j,k}^{\ c} + T_{b\ k}^{\ c}T_{i\ j}^{\ b} + S_{i\ jk}^{\ c} = 0$ or equivalently $T_{ijk,\ell} + M_{j\ell ik} + S_{ijk\ell} = 0$. They satisfy linear constraints determined by the conditions on $T_{j\ k}^{\ i}$, namely

(1) $S_{i\ jk}^{\ c} + S_{j\ ik}^{\ c} = 0$ (Antisymmetry) (2) $S_{[i\ jk]}^{\ c} = 0$ (Jacobi) (3) $S^{i}_{\ [kj\ell]} - \frac{1}{2} S_{[kj\ell]}^{\ \ \ \ \ i} = 0$ (Cyclic)

and as we remarked before by lowering the i index on the cyclic identity (3) we can see it is implied by (1) and (2). In particular then for any scalars $P_{j\ k\ell}^{\ i}$, n^4 in number, we have that $S_{j\ k\ell}^{\ i} = P_{[j\ k]\ell}^{\ \ i} - P_{[j\ k\ell]}^{\ \ i}$ satisfies the required conditions (1) through (3) for $S_{j\ k\ell}^{\ i}$ and gives us the form for the symmetry projection onto the vector space of all $S_{j\ k\ell}^{\ i}$ satisfying the required conditions. Furthermore, as required of any projection, it equals the identity when restricted to its image vector space. This makes it possible, within the constraints of the Jacobi identity, to reformulate the geometry structure scalars in order to simplify the expression for the Riemann tensor. This will be done in the next section.

In closing, we should remark that the integrability condition for $T_{j\ k}^{\ i}$ can be used to give us an important equation in the geometry structure scalars. If we differentiate the defining equation for $S_{i\ jk}^{\ c}$ with respect to direction ℓ, antisymmetrize in k and ℓ and use integrability for $T_{i\ j}^{\ c}$ we find that

$$\begin{aligned} 0 = T_{i\ j}^{\ e}T_{d\ e}^{\ c}T_{k\ \ell}^{\ d} + 2T_{i\ j}^{\ b}S_{b\ [\ell k]}^{\ c} + 2S_{i\ j[k,\ell]}^{\ c} \\ + 2S_{i\ j[\ell}^{\ b}T_{k]\ b}^{\ \ c} - S_{i\ jd}^{\ c}T_{k\ \ell}^{\ d}. \end{aligned} \qquad \text{(III.7.1)}$$

(III.8) Reduced Geometry Structure Scalars

Let us write down a summary of the contractions of $T_{j\ k}^{\ i}$ and the identities that hold.

(III.8.1) Contractions of $T_{j\ k}^{\ i}$

$M_{ikj\ell} = T_{aik}T_{j\ \ell}^{\ a} (M_{ikj\ell} + M_{ik\ell j} = 0)$

$$F_{ik\ell j} = T_{aik}T^{a}{}_{\ell j} \quad (F_{ik\ell j} = F_{\ell jik})$$

$$C_{ikj\ell} = T_{iak}T_{j}{}^{a}{}_{\ell} \quad (C_{ikj\ell} = -C_{kij\ell} = -C_{ik\ell j} = C_{j\ell ik}).$$

(III.8.2) Double Contractions of $T_{j}{}^{i}{}_{k}$ $\quad (T_a = T_{a}{}^{b}{}_{b})$

$$A_{kj} = T_a T^{a}{}_{kj},\ D_{jk} = T^{i}{}_{ja}T_{i}{}^{a}{}_{k}$$

$$B_{jk} = T^{i}{}_{aj}T_{i}{}^{a}{}_{k} \ (B_{jk} = B_{kj})$$

$$E_{kj} = T_{ak}{}^{i}T^{a}{}_{ji} \ (E_{kj} = E_{jk})$$

$$G_{kj} = T_{a}{}^{i}{}_{k}T_{i}{}^{a}{}_{j} \ (G_{kj} = G_{jk})$$

(III.8.3) Contraction Identities

$$M^{i}{}_{ki\ell} = G_{k\ell},\ M_{i}{}^{k}{}_{k\ell} = -D_{i\ell},\quad F^{i}{}_{i\ell j} = F_{\ell j}{}^{i}{}_{i} = A_{\ell j},$$

$$F^{i}{}_{kij} = B_{kj},\ F^{i}{}_{kji} = F_{j}{}^{i}{}_{ik} = D_{jk},\ F_{k}{}^{i}{}_{ji} = E_{kj},$$

$$C^{i}{}_{ki\ell} = B_{k\ell},\ M^{i}{}_{ijk} = -M^{i}{}_{ikj}.$$

The linear projection function is $P_{j}{}^{i}{}_{k\ell} \to S_{j}{}^{i}{}_{k\ell} = P_{[j}{}^{i}{}_{k]\ell} - P_{[j}{}^{i}{}_{k\ell]}$ so $S_{j}{}^{i}{}_{k\ell} = \frac{1}{3}\left(P_{j}{}^{i}{}_{k\ell} - P_{k}{}^{i}{}_{j\ell}\right) + \frac{1}{6}\left(P_{j}{}^{i}{}_{\ell k} + P_{\ell}{}^{i}{}_{kj} - P_{k}{}^{i}{}_{\ell j} - P_{\ell}{}^{i}{}_{jk}\right)$. By replacing the derivative of $T_{j}{}^{i}{}_{k}$ terms in the Riemann tensor expression (III.4.1) by M terms and S terms using the differential equation for T we have

$$\begin{aligned} R_{ijk\ell} = {} & \tfrac{1}{2}\left(-S_{ki\ell j} + S_{kj\ell i} - S_{ikj\ell} + S_{i\ell jk}\right) - \tfrac{1}{2}C_{ijk\ell} \\ & + \tfrac{1}{4}\left(C_{i\ell jk} - C_{ikj\ell}\right) + \tfrac{1}{4}\left(F_{i\ell kj} + F_{i\ell jk} + F_{\ell ikj} + F_{\ell ijk}\right) \\ & - \tfrac{1}{4}\left(F_{ik\ell j} + F_{ikj\ell} + F_{ki\ell j} + F_{kij\ell}\right) + \tfrac{1}{4}\left(M_{kij\ell} - M_{ikj\ell}\right. \\ & \left. + M_{\ell jik} - M_{j\ell ik}\right) - \tfrac{1}{4}\left(M_{\ell ijk} - M_{i\ell jk} + M_{kji\ell} - M_{jki\ell}\right) \\ & + \tfrac{1}{2}\left(-M_{ijk\ell} + M_{jik\ell} - M_{k\ell ij} + M_{\ell kij}\right). \end{aligned}$$

In the expression above for $R_{ijk\ell}$ the part involving the four S terms only satisfies all the symmetries of the Riemann tensor, namely $R_{ijk\ell} = R_{k\ell ij} = -R_{jik\ell} = -R_{ij\ell k}$ and $R_{i[jk\ell]} = 0$. The same holds true for the parts of the expression involving only C terms, F terms or M terms.

We may write $S_{j\ k\ell}^{\ i} = \check{S}_{j\ k\ell}^{\ i} + \tilde{S}_{j\ k\ell}^{\ i}$ where $P_{j\ k\ell}^{\ i} = \check{P}_{j\ k\ell}^{\ i} + \tilde{P}_{j\ k\ell}^{\ i}$ and $S_{j\ k\ell}^{\ i} = P_{[j\ k]\ell}^{\ \ i} - P_{[j\ k\ell]}^{\ \ i}$, $\check{S}_{j\ k\ell}^{\ i} = \check{P}_{[j\ k]\ell}^{\ \ i} - \check{P}_{[j\ k\ell]}^{\ \ i}$ and $\tilde{S}_{j\ k\ell}^{\ i} = \tilde{P}_{[j\ k]\ell}^{\ \ i} - \tilde{P}_{[j\ k\ell]}^{\ \ i}$. Remarkably enough if we choose $\check{P}_{ijk\ell} = -\frac{3}{4} C_{ijk\ell} - \frac{1}{2} F_{ijk\ell} - \frac{1}{2} F_{ij\ell k} - 3M_{ijk\ell}$ and substitute, all the terms involving C, F and M cancel out of the expression for the Riemann tensor and we obtain

$$R_{ijk\ell} = \frac{1}{2}\left(-\tilde{S}_{ki\ell j} + \tilde{S}_{kj\ell i} - \tilde{S}_{ikj\ell} + \tilde{S}_{i\ell jk}\right). \qquad \text{(III.8.4)}$$

If we write out the differential equation for $T_{j\ k}^{\ i}$ now in terms of $\tilde{S}$ we have $T_{jik,\ell} + M_{i\ell jk} + \check{S}_{jik\ell} + \tilde{S}_{jik\ell} = 0$, and substituting the full expression for $\check{S}_{jik\ell}$ we have

$$T_{jik,\ell} + M_{i\ell jk} + \frac{1}{8}\left(C_{ji\ell k} + C_{kij\ell} + 2C_{\ell ijk}\right) + \frac{1}{4}\left(F_{kij\ell} + F_{ki\ell j} - F_{jik\ell} - F_{ji\ell k}\right) + \frac{1}{2}\left(M_{kij\ell} - M_{jik\ell} + 2M_{\ell ijk}\right) + \tilde{S}_{jik\ell} = 0. \qquad \text{(III.8.5)}$$

Thus we may specify $\tilde{S}_{jik\ell}$ satisfying the symmetry conditions for S, and $R_{ijk\ell}$ (in frame components) is very simply determined from it using (III.8.4). If $T_{j\ k}^{\ i}$ is then specified independently $(T_{j\ k}^{\ i} = -T_{k\ j}^{\ i})$ then $T_{j\ k,\ell}^{\ i}$ is determined by (III.8.5) with the Jacobi identity satisfied. This can be done arbitrarily at any one point, and in neighborhoods is subject to the integrability condition for T. The expressions in

the differential equation (III.8.5) for $T_{j\ k}^{\ i}$ above involving C only, F only or M only each satisfy the symmetry conditions on S where it is understood the term $M_{i\ell jk}$ is deleted from the M term expression.

We may write $\dot{S}_{i\ k\ell}^{\ j} = -R_{[i\ k]\ell}^{\ \ j} + R_{[i\ k\ell]}^{\ \ j} = -\frac{1}{2} R^{j}_{\ \ell ik}$ so $\dot{S}_{ijk\ell} = \frac{1}{4}\left(\tilde{S}_{ji\ell k} + \tilde{S}_{ijk\ell} + \tilde{S}_{\ell kji} + \tilde{S}_{k\ell ij}\right)$. Then putting $\overset{*}{S}_{ijk\ell} = \tilde{S}_{ijk\ell} - \dot{S}_{ijk\ell}$ we see that $\overset{*}{S}$ satisfies the usual symmetries of S as well as the additional symmetry $\overset{*}{S}_{ji\ell k} + \overset{*}{S}_{ijk\ell} + \overset{*}{S}_{\ell kji} + \overset{*}{S}_{k\ell ij} = 0$ and therefore the dependence on the Riemann tensor in (III.8.5) can be factored out to give,

$$T_{jik,\ell} + M_{i\ell jk} + \frac{1}{8}\left(C_{ji\ell k} + C_{kij\ell} + 2C_{\ell ijk}\right) + \frac{1}{4}\left(F_{kij\ell} + F_{ki\ell j} - F_{jik\ell} - F_{ji\ell k}\right) + \frac{1}{2}\left(M_{kij\ell} - M_{jik\ell} + 2M_{\ell ijk}\right) - \frac{1}{2} R_{i\ell jk} + \overset{*}{S}_{jik\ell} = 0.$$

Of course $\overset{*}{S}_{ijk\ell} = \frac{3}{4}\tilde{S}_{ijk\ell} - \frac{1}{4}\left(\tilde{S}_{ji\ell k} + \tilde{S}_{\ell kji} + \tilde{S}_{k\ell ij}\right)$ and this is independent of the Riemann Tensor. The $\tilde{S}$ scalars are called the *reduced geometry structure scalars*, and $\overset{*}{S}$ are the *reduced geometry structure scalars for flat space-time*, $\overset{*}{S}_{jik\ell} = \frac{1}{2} R_{i\ell jk} + \tilde{S}_{jik\ell}$. The Ricci Tensor, given by

$$R_{jk} = \frac{1}{2} B_{jk} - \frac{1}{4} E_{jk} + \frac{3}{2} G_{jk} + A_{(jk)} + D_{(jk)} - S^{i}_{\ (jk)i} - S_{(k\ \ j)}^{\ \ i\ell}\eta_{i\ell}$$

can also be simplified greatly by the use of reduced scalars.

(III.9) Specific Examples and Physical Interpretation

Let us look first at the case where $S_{i\ jk}^{\ c} = 0$ identically on U_α so $T_{i\ j}^{\ e}T_{d\ e}^{\ c}T_{k\ \ell}^{\ d} = 0$ (II.7.1) from integrability and $T_{i\ j,k}^{\ c} + T_{b\ k}^{\ c}T_{i\ j}^{\ b} = 0$. The largest class of simple solutions to (II.7.1) are the *factored solutions* of the form $T_{j\ k}^{\ i} = v^{i}\Lambda_{jk}$ where $\Lambda_{jk} + \Lambda_{kj} = 0$. Then $T_{i\ j,k}^{\ c} + v^{c}\Lambda_{bk}v^{b}\Lambda_{ij} = 0 = T_{i\ j,k}^{\ c} - T_{k}T_{i\ j}^{\ c}$ where $T_k = T_{k\ i}^{\ i}$. We may

write $(\ln T_{i\ j}^{\ c})_{,k} = T_k$ so $T_{i\ j}^{\ c} = f\tilde{T}_{i\ j}^{\ c}$ where $\tilde{T}_{i\ j}^{\ c}$ is a constant, i.e. $\tilde{T}_{i\ j,k}^{\ c} = 0$, $\tilde{T}_{i\ j}^{\ c} = \tilde{v}^c\tilde{\Lambda}_{ij}$, $\tilde{v}^c_{\ ,i} = 0$, $\tilde{\Lambda}_{ij,k} = 0$. Thus $(\ln f)_{,k} = T_k = f\tilde{T}_k$ where $\tilde{T}_k = \tilde{T}_{k\ i}^{\ i} = \tilde{v}^i\tilde{\Lambda}_{ki}$ is a constant. Hence $f_{,k} = f^2\tilde{T}_k$ implies $(-\frac{1}{f})_{,k} = \tilde{T}_k$. Let $\hat{T} = -\frac{1}{f}$ so that $\tilde{T}_k = \hat{T}_{,k}$, i.e. $\hat{T}$ is a scalar field which has all derivatives constant. Clearly $\hat{T}_{,k\ell} = 0$ and this is consistent with $\hat{T}_{,k\ell} - \hat{T}_{,\ell k} = f\tilde{T}_{k\ \ell}^{\ i}\tilde{T}_i = f\tilde{v}^i\tilde{\Lambda}_{k\ell}\tilde{\Lambda}_{ij}\tilde{v}^j = 0$ by antisymmetry of $\tilde{\Lambda}_{ij}$.

Let us look at the Ricci tensor in this case. We have $R_{jk} = \frac{1}{2} B_{jk} - \frac{1}{4} E_{jk} + \frac{3}{2} G_{jk} + A_{(jk)} + D_{(jk)}$ since $S_{i\ jk}^{\ c} = 0$ and these terms are defined in (III.8.2). We now substitute in the expression $T_{b\ c}^{\ a} = f\tilde{v}^a\tilde{\Lambda}_{bc}$, $f_{,\ell} = fT_\ell = fT_{\ell\ a}^{\ a}$ and put $\tilde{v}^i = \bar{\lambda}u^i + h^i$, $\tilde{\Lambda}_{ik} = u_i g_k - u_k g_i + \epsilon_{ik\ell}w^\ell$ where $\underset{\sim}{g}$, $\underset{\sim}{h}$, $\underset{\sim}{w}$ are constant vectors in $\mathbb{R}^3$ (i.e. have zero 4 component in these adapted frame components). This is the most general form of the factored solution. Using the usual notation for dot, cross product, and vector length squared in $\mathbb{R}^3$ we find that

$$\begin{aligned}
\frac{1}{f^2} R_{jk} = {}& \left(2\bar{\lambda}^2 - \frac{h^2}{2}\right) g_j g_k + \frac{1}{2}(h^2-\bar{\lambda}^2)w^2\gamma_{jk} - \frac{1}{2}(h^2-\bar{\lambda}^2)w_j w_k + \frac{1}{2}(3w^2+g^2)h_j h_k \\
& + \frac{3}{2}\bar{\lambda}[g_j(\underset{\sim}{h}\times\underset{\sim}{w})_k + (\underset{\sim}{h}\times\underset{\sim}{w})_j g_k] + \frac{3}{2}(\underset{\sim}{h}\times\underset{\sim}{w})_j(\underset{\sim}{h}\times\underset{\sim}{w})_k - \lambda[h_j(\underset{\sim}{g}\times\underset{\sim}{w})_k + (\underset{\sim}{g}\times\underset{\sim}{w})_j h_k] \\
& - (\underset{\sim}{h}\cdot\underset{\sim}{g})[h_k g_j + h_j g_k] - (\underset{\sim}{h}\cdot\underset{\sim}{w})[h_j w_k + h_k w_j] \\
& + u_j\left\{\frac{1}{2}(h^2-3\bar{\lambda}^2)(\underset{\sim}{g}\times\underset{\sim}{w})_k + \left[\frac{\bar{\lambda}}{2}(w^2-g^2) - \underset{\sim}{g}\cdot\underset{\sim}{h}\times\underset{\sim}{w}\right]h_k + \frac{3}{2}(\underset{\sim}{h}\cdot\underset{\sim}{g})(\underset{\sim}{h}\times\underset{\sim}{w})_k \right. \\
& \left. + \frac{1}{2}\bar{\lambda}(\underset{\sim}{h}\cdot\underset{\sim}{g})g_k - \bar{\lambda}(\underset{\sim}{h}\cdot\underset{\sim}{w})w_k\right\} \\
& + u_k\left\{\frac{1}{2}(h^2-3\bar{\lambda}^2)(\underset{\sim}{g}\times\underset{\sim}{w})_j + \left[\frac{\bar{\lambda}}{2}(w^2-g^2) - \underset{\sim}{g}\cdot\underset{\sim}{h}\times\underset{\sim}{w}\right]h_j + \frac{3}{2}(\underset{\sim}{h}\cdot\underset{\sim}{g})(\underset{\sim}{h}\times\underset{\sim}{w})_j \right. \\
& \left. + \frac{1}{2}\bar{\lambda}(\underset{\sim}{h}\cdot\underset{\sim}{g})g_j - \bar{\lambda}(\underset{\sim}{h}\cdot\underset{\sim}{w})w_j\right\} \\
& + u_j u_k\left[\left(\frac{h^2}{2} - 2\bar{\lambda}^2\right)g^2 - \frac{\bar{\lambda}^2}{2}w^2 + \frac{3}{2}(\underset{\sim}{h}\cdot\underset{\sim}{g})^2 - 2\bar{\lambda}\underset{\sim}{g}\cdot(\underset{\sim}{h}\times\underset{\sim}{w})\right] . \qquad \text{(III.9.1)}
\end{aligned}$$

Here we see that R_{jk} is simply f^2 times a constant matrix. The field equations tell us that (with $\kappa = 8\pi G/c^4$),

$$\sigma_1 = -\frac{1}{2\kappa}\left(R_{33}+R_{22}-R_{11}-R_{44}\right), \quad \sigma_2 = -\frac{1}{2\kappa}\left(R_{33}+R_{11}-R_{22}-R_{44}\right)$$

$$\sigma_3 = -\frac{1}{2\kappa}\left(R_{11}+R_{22}-R_{33}-R_{44}\right), \quad \rho c^2+\epsilon = -\frac{1}{2\kappa}\left(R_{11}+R_{22}+R_{33}+R_{44}\right)$$

$$\lambda\alpha^i = \frac{1}{\kappa} R_{i4}, \quad (\alpha^1)^2+(\alpha^2)^2+(\alpha^3)^2=1. \tag{III.9.2}$$

Of course if we are to be working in adapted frame components then T_{ij} (and hence R_{ij}) must be diagonal in the orthogonal part. One way of achieving this in (III.9.1) is to take $h_a = hr_a$, $g_a = gs_a$, $w_a = wt_a$ where $r_a = \delta_{a1}$, $s_a = \delta_{a2}$, $t_a = \delta_{a3}$. Doing this we find that

$$\sigma_1 = -\frac{f^2}{2\kappa}\left(4\bar{\lambda}^2g^2 - 3\bar{\lambda}ghw - \frac{3}{2}g^2h^2 + \frac{\bar{\lambda}^2w^2}{2}\right)$$

$$\sigma_2 = -\frac{f^2}{2\kappa}\left(\frac{g^2h^2}{2} + \frac{\bar{\lambda}^2w^2}{2} - \bar{\lambda}ghw\right)$$

$$\sigma_3 = -\frac{f^2}{2\kappa}\left(4\bar{\lambda}^2g^2 - \frac{h^2g^2}{2} - \frac{\bar{\lambda}^2w^2}{2} + 4h^2w^2 - 7\bar{\lambda}ghw\right)$$

$$\rho c^2+\epsilon = -\frac{f^2}{2\kappa}\left(\frac{g^2h^2}{2} - 3\bar{\lambda}ghw + 4h^2w^2 - \frac{3\bar{\lambda}^2w^2}{2}\right),$$

$\alpha^1 = 1$, $\alpha^2 = \alpha^3 = 0$, and

$$\lambda = \frac{f^2}{\kappa}\left[-\frac{1}{2}(h^2-3\bar{\lambda}^2)gw - \left(\frac{\bar{\lambda}}{2}(w^2-g^2) + ghw\right)h\right]. \tag{III.9.3}$$

Of course we must have $\rho c^2+\epsilon \geq 2\lambda+\sigma_{max}$. We can satisfy this inequality strictly if $h = 0$ and $w^2 > 6|gw| + 4g^2$, and by continuity it must hold for all $(h,g,w) \in \mathbb{R}^3$ in a neighborhood of such a point. Of course $\bar{\lambda}$, h, g, w are constants in any one solution to the field equations and space and time dependence is only in f. With $r_a = r^a = \delta_{a1}$, $s_a = s^a = \delta_{a2}$, $t^a = t_a = \delta_{a3}$, $u^a = -u_a = \delta_{a4}$ we have, in this special case

$$T_b{}^a{}_c = f(\bar{\lambda}u^a + hr^a)[g(u_b s_c - u_c s_b) + w(r_b s_c - r_c s_b)]$$

and $T_b = T_b{}^a{}_a = f(\bar{\lambda}g - hw)s_b$. Therefore, $f_{,2} = f^2(\bar{\lambda}g - hw)$ and $f_{,1} = f_{,3} = f_{,4} = 0$. Using this we see that from the relations

$$\omega_{AB} = \frac{1}{2} T_A{}^4{}_B, \quad \theta_{AB} = -\frac{1}{2}\left(T^4{}_{BA} + T^4{}_{AB}\right)$$

$$\theta = T_4 = T_4{}^I{}_I, \quad \dot{u}_a = T_a{}^4{}_4 \tag{III.9.4}$$

that $\omega_{12} = \frac{1}{2} f\bar{\lambda}w = -\omega_{21}$, $\omega_{13} = \omega_{23} = \omega_{31} = \omega_{32} = 0$, i.e. $\upsilon^I = -\frac{1}{2} f\bar{\lambda}wt^I$, $\theta_{AB} = -\frac{1}{2} fgh(r_A s_B + r_B s_A)$, $\theta = 0$, $\dot{u}_A = fg\bar{\lambda}s_A$. Notice that $\hat{T}^{AB}\theta_{AB} = 0$ since we are working in adapted frame components. Hence we have an isochoric motion in which the stresses do no work on the system and for which the vorticity and acceleration are orthogonal to one another and along two of the principal axes of stress.

Of course we can check the equations (III.9.3) by using the energy and momentum conservation equations, namely the ones we derived in (III.5) i.e.,

$$T_4{}^I{}_I\sigma_I = \epsilon T_4{}^I{}_I + \epsilon_{,4} + (\lambda\alpha^I)_{,I} + \lambda\alpha^I\left(T_I{}^b{}_b + T_I{}^4{}_4\right),$$

$$(\rho c^2 + \epsilon)T_A{}^4{}_4 = \sigma_{A,A} + \sigma_A T_A{}^b{}_b - \sigma_I T_A{}^I{}_I \tag{III.9.5}$$

$$- \left[\lambda\left(T_A{}^4{}_B + T_4{}^B{}_A\right)\alpha^B + \left(\lambda_{,4} + \lambda T_4{}^I{}_I\right)\alpha^A + \lambda\alpha^A{}_{,4}\right]$$

and substituting in. This is rather tedious in general, but has been done for specific examples.

One other case of interest where (III.9.1) is diagonal in the orthogonal part and therefore represents the Ricci tensor in an adapted frame component system is the case $h_a = hr_a$, $g_a = gr_a$, $w_a = wr_a$. Then $\frac{1}{f^2} R_{jk} = \frac{1}{2}(h^2 - \bar{\lambda}^2)w^2\gamma_{jk} + \beta r_j r_k - \frac{\bar{\lambda}}{2} w^2 h(u_j r_k + u_k r_j) - (h^2 w^2 + \beta)u_j u_k$

where $\beta = 2\bar{\lambda}^2 g^2 + \frac{1}{2}\bar{\lambda}^2 w^2 - h^2 w^2 - 2h^2 g^2$. Of course $\gamma_{jk} = r_j r_k + s_j s_k + t_j t_k$ and has the components of the 3×3 identity matrix $\gamma_{jk} = \delta_{jk}$ in adapted coordinates. Using (III.9.2) we find that

$$\sigma_1 = \frac{f^2 w^2}{4\kappa}(\bar{\lambda}^2 - 3h^2), \quad \sigma_2 = \sigma_3 = \frac{f^2}{\kappa}(h^2 - \bar{\lambda}^2)\left(\frac{w^2}{4} + 2g^2\right),$$

$$\rho c^2 + \epsilon = \frac{f^2 w^2}{4\kappa}(3\bar{\lambda}^2 - h^2), \quad \alpha^1 = 1, \quad \alpha^2 = \alpha^3 = 0, \quad \lambda = \frac{f^2 w^2 \bar{\lambda} h}{2\kappa}. \tag{III.9.6}$$

In particular the condition $\rho c^2 + \epsilon \geq 2|\lambda| + \sigma_{max}$ implies that for a nontrivial solution, $\beta \neq 0$. It is clear that if we consider $\bar{\lambda} \neq 0$, $h = g = 0$, $w \neq 0$ then we have the condition on the energy momentum tensor holding with strict inequality, and therefore by continuity it holds for values of these parameters in a neighborhood of such a point (in parameter space).

In this case we have $T_b{}^a{}_c = f \cdot (\bar{\lambda}u^a + hr^a)(2gu_{[b}r_{c]} + 2ws_{[b}t_{c]})$ and so $T_b = fg(\bar{\lambda}r_b + hu_b)$. Therefore $f_{,\ell} = fT_\ell = f^2 g(\bar{\lambda}r_\ell + hu_\ell)$ so $f_{,4} = -f^2 gh$, $f_{,1} = f^2 g\bar{\lambda}$, $f_{,2} = f_{,3} = 0$. If we write $f_{,\ell} = \dot{f}u_\ell + f'r_\ell$ then $f' = f^2 g\bar{\lambda}$ and $\dot{f} = f^2 gh$ since $u_4 = -1$. This is the notation we will use in the section on unidirectional space times later in this chapter, and in fact the solution we are considering here is in fact unidirectional and a special case of the more general result (III.15).

Now we can see (using (III.9.4)) that $\omega_{23} = \frac{1}{2} f\bar{\lambda}w$, $\upsilon^1 = -\frac{1}{2} f\bar{\lambda}w$, $\upsilon^2 = \upsilon^3 = 0$, $\theta_{AB} = -fghr_A r_B$, $\theta = -fgh$, $\dot{u}_a = fg\bar{\lambda}r_a$. In this motion, vorticity and acceleration are parallel, and along the same principal axis.

In these solutions to the field equations of relativity, we have not introduced a constitutive equation at this time specifying a

relationship between deformation (or its history) and the stress. This, of course, puts further limits on the possible choice of the free parameters specified, which are constants in any one solution but arbitrary over a range of problems subject to the positive energy density condition on the energy momentum tensor, namely $\rho c^2 + \epsilon \geq 2\lambda + \sigma_{max}$. The solutions, in this section, to the Einstein equations are of a very special and limited type that are given to illustrate the technique of solving and interpreting physically using the Ricci coefficients. More general and physically interesting solutions will come later.

(III.10) Transformation into Homogeneous Coordinates.

It is of interest to be able to transform solutions we have in a metric frame component system over to a homogeneous coordinate system with a line element $ds^2 = g_{ij}dx^i dx^j$. Let $\underset{\sim}{v}_{(a)}$, $a = 1,\cdots,n$ be n vector fields determining a frame component system in an open neighborhood U_α of some x in a differential manifold M of dimension n. Let γ identify a homogeneous coordinate system about $x \in U_\gamma$ and let $v^i_{(a)}$ $i = 1,\cdots,n$ be the n coordinate components of $\underset{\sim}{v}_{(a)}$ in the γ system. We let α be the index of the frame components determined by $\underset{\sim}{v}_{(a)}$ and let $T_b{}^a{}_c$ denote the Ricci rotation coefficients. We let $T_b{}^a{}_{c\natural i}$ denote the γ coordinate partial derivative, and $T_b{}^a{}_{c,d}$ the frame component derivative. For the sake of this discussion here we will use indices i, j, k, etc. for the γ system and a, b, c, etc. for the α system. Clearly $v^i_{(a)} = G_{\gamma\alpha}{}^i{}_a$ is invertible, i.e. there exists $G_{\alpha\gamma}{}^b{}_j$ with $G_{\gamma\alpha}{}^i{}_a G_{\alpha\gamma}{}^a{}_j = \delta^i_j$, $G_{\gamma\alpha}{}^i{}_a G_{\alpha\gamma}{}^b{}_i = \delta^b_a$.

In specifying conditions for the existence of $v^i_{(a)}$ on a neighborhood we see that if it is invertible at one point it will be on an open set containing that point by continuity. Of course we have $T_b{}^a{}_{c,d} =$

$T_{b}{}^{a}{}_{c\text{ϟ}i}v^{i}{}_{(d)}$ and require the Jacobi identity $T_{[b}{}^{a}{}_{c,d]} + T_{e}{}^{a}{}_{[b}T_{c}{}^{e}{}_{d]} = 0$. The basic Lie bracket condition $[\underset{\sim}{v}_{(a)}, \underset{\sim}{v}_{(b)}] = T_{b}{}^{c}{}_{a}\underset{\sim}{v}_{(c)}$ can be written as $v^{i}{}_{(b),a} - v^{i}{}_{(a),b} = T_{b}{}^{c}{}_{a}v^{i}{}_{(c)}$. This is simply the integrability condition for v^{i} where $v^{i}{}_{(b)} = v^{i}{}_{,b}$ and v^{i} are the γ-coordinate system scalar functions defined on $U_{\gamma} \subset M$. In terms of homogeneous derivatives $v^{i}{}_{(b)\text{ϟ}j}v^{j}{}_{(a)} - v^{i}{}_{(a)\text{ϟ}j}v^{j}{}_{(b)} = T_{b}{}^{c}{}_{a}v^{i}{}_{(c)}$. We have that $v^{i}{}_{(b),a} + v^{i}{}_{(a),b} = Y_{b}{}^{c}{}_{a}v^{i}{}_{(c)}$ for some scalars $Y_{b}{}^{c}{}_{a} = Y_{a}{}^{c}{}_{b}$. Then $v^{i}{}_{(b),a} = \frac{1}{2}\left(T_{b}{}^{c}{}_{a} + Y_{b}{}^{c}{}_{a}\right)v^{i}{}_{(c)}$. This permits us to obtain $v^{i}{}_{(b)\text{ϟ}j}$ by multiplying by the inverse of $v^{j}{}_{(a)}$ and then the condition $v^{i}{}_{(b)\text{ϟ}jk} = v^{i}{}_{(b)\text{ϟ}kj}$ of local integrability can be imposed. Equivalently we can impose the condition $v^{i}{}_{(b),ae} - v^{i}{}_{(b),ea} = T_{a}{}^{d}{}_{e}v^{i}{}_{(b),d}$ directly for integrability. We see

$$v^{i}{}_{(b),ae} = \frac{1}{2}\left(T_{b}{}^{c}{}_{a,e} + Y_{b}{}^{c}{}_{a,e}\right)v^{i}{}_{(c)} + \frac{1}{2}\left(T_{b}{}^{c}{}_{a} + Y_{b}{}^{c}{}_{a}\right)v^{i}{}_{(c),e}$$

where $v^{i}{}_{(c),e} = \frac{1}{2}\left(T_{c}{}^{d}{}_{e} + Y_{c}{}^{d}{}_{e}\right)v^{i}{}_{(d)}$. If we substitute this above, evaluate $v^{i}{}_{(b),ae} - v^{i}{}_{(b),ea}$ and equate this to $\frac{1}{2}T_{a}{}^{d}{}_{e}\left(T_{b}{}^{c}{}_{d} + Y_{b}{}^{c}{}_{d}\right)v^{i}{}_{(c)}$ and simplify, we can eliminate all reference to $v^{i}{}_{(c)}$ and obtain, using the Jacobi identity, the geometry structure scalars in the form

$$\begin{aligned} S_{a}{}^{d}{}_{eb} = {} & T_{a}{}^{c}{}_{e}T_{b}{}^{d}{}_{c} + \frac{1}{2}T_{c}{}^{d}{}_{a}T_{e}{}^{c}{}_{b} + \frac{1}{2}T_{c}{}^{d}{}_{e}T_{b}{}^{c}{}_{a} + Y_{b}{}^{d}{}_{e,a} - Y_{b}{}^{d}{}_{a,e} \\ & + T_{a}{}^{c}{}_{e}Y_{b}{}^{d}{}_{c} - \frac{1}{2}T_{b}{}^{c}{}_{a}Y_{c}{}^{d}{}_{e} + \frac{1}{2}T_{b}{}^{c}{}_{e}Y_{c}{}^{d}{}_{a} - \frac{1}{2}Y_{b}{}^{c}{}_{a}T_{c}{}^{d}{}_{e} + \frac{1}{2}Y_{b}{}^{c}{}_{e}T_{c}{}^{d}{}_{a} \\ & - \frac{1}{2}Y_{b}{}^{c}{}_{a}Y_{c}{}^{d}{}_{e} + \frac{1}{2}Y_{b}{}^{c}{}_{e}Y_{c}{}^{d}{}_{a} . \end{aligned} \tag{III.10.1}$$

If we find a smooth field of scalars $Y_{c}{}^{d}{}_{a} = Y_{a}{}^{d}{}_{c}$ satisfying this equation, we have a direct differential equation, namely $v^{i}{}_{(b),a} = \frac{1}{2}\left(T_{b}{}^{c}{}_{a} + Y_{b}{}^{c}{}_{a}\right)v^{i}{}_{(c)}$ for the frame component to coordinate transfer coefficients $v^{i}{}_{(c)}$. In (III.10.1) $S_{a}{}^{d}{}_{eb}$ automatically satisfies the

required symmetry conditions. This equation, a consequence of the integrability condition of $v^i_{(c)}$ gives us an extra condition on $Y_b{}^d{}_{[e,a]}$ we would not otherwise have.

In the special solutions of (III.9) we had $S_i{}^c{}_{jk} \equiv 0$ and $T_j{}^i{}_k = f\,\tilde{v}^i\,\tilde{\Lambda}_{jk}$ where $\tilde{v}^i$ and $\tilde{\Lambda}_{jk}$ were constants, and $f_{,k} = f^2\tilde{T}_k$ and $\tilde{T}_k = \tilde{v}^i\tilde{\Lambda}_{ki}$. As we saw, this satisfied the Jacobi identity and integrability conditions for f. It is easy to see that (III.10.1) holds for $Y_b{}^d{}_c = f\tilde{Y}_b{}^d{}_c$ where $\tilde{Y}_b{}^d{}_c = -\dfrac{\tilde{v}^d}{(\tilde{v})^2}(\tilde{v}_b\tilde{T}_c + \tilde{v}_c\tilde{T}_b)$ where $(\tilde{v})^2 = \tilde{v}_a\tilde{v}^a$ is non-zero (i.e. $\tilde{v}$ is non-null). Using this we can solve the differential equation

$$v^i_{(b),a} = \frac{1}{2}\left(T_b{}^c{}_a + Y_b{}^c{}_a\right)v^i_{(c)} \tag{III.10.2}$$

for $v^i_{(c)}$ taking say $v^i_{(c)} = \delta^i_c$, $i,c = 1,2,3,4$ at $x \in M$ and working in a neighborhood of x. We will illustrate this technique in (III.14) when we discuss the omnidirectional space-times and obtain, using this method, a Friedmann metric from one of the omnidirectional solutions.

The equations (III.10.1) can be split up into a form that can more easily be handled. We may write it as two equations, namely

$$S_a{}^d{}_{eb} = -\bar{Q}_b{}^d{}_{ea} + \bar{Q}_b{}^d{}_{ae}, \quad \text{and}$$

$$0 = Y_b{}^d{}_{e,a} + \frac{1}{2}\dot{Y}_b{}^c{}_e W_c{}^d{}_a + \frac{1}{2}T_a{}^c{}_e W_b{}^d{}_c + \frac{1}{2}T_a{}^c{}_b W_e{}^d{}_c + \bar{Q}_b{}^d{}_{ea} \tag{III.10.3}$$

where $W_b{}^a{}_c = Y_b{}^a{}_c + T_b{}^a{}_c$ and $\bar{Q}_b{}^d{}_{ea}$ has the only symmetry $\bar{Q}_b{}^d{}_{ea} = \bar{Q}_e{}^d{}_{ba}$. Therefore we also have $0 = T_b{}^d{}_{e,a} + T_c{}^d{}_a T_b{}^c{}_e + \bar{Q}_a{}^d{}_{be} - \bar{Q}_a{}^d{}_{eb}$ and adding we obtain

$$\begin{aligned} 0 = W_b{}^d{}_{e,a} &+ \frac{1}{2}Y_b{}^c{}_e W_c{}^d{}_a + \frac{1}{2}T_a{}^c{}_e W_b{}^d{}_c + \frac{1}{2}T_a{}^c{}_b W_e{}^d{}_c \\ &+ T_c{}^d{}_a T_b{}^c{}_e + \bar{Q}_b{}^d{}_{ea} + \bar{Q}_a{}^d{}_{be} - \bar{Q}_a{}^d{}_{eb}\,. \end{aligned} \tag{III.10.4}$$

The equations could also be written in the form

$$0 = Y_{b\ e,a}^{\ d} + \frac{1}{2} Y_{b\ e}^{\ c} Y_{c\ a}^{\ d} + \frac{1}{2} Y_{b\ e}^{\ c} T_{c\ a}^{\ d} + \frac{1}{2} T_{a\ e}^{\ c} Y_{b\ c}^{\ d} - \frac{1}{2} T_{b\ a}^{\ c} Y_{c\ e}^{\ d} + Q_{b\ ea}^{\ d} ,$$

where $S_{a\ eb}^{\ d} = T_{a\ e}^{\ c} T_{b\ c}^{\ d} + \frac{1}{2} T_{c\ a}^{\ d} T_{e\ b}^{\ c} + \frac{1}{2} T_{c\ e}^{\ d} T_{b\ a}^{\ c} + \mathring{S}_{a\ eb}^{\ d}$ with $\mathring{S}_{a\ eb}^{\ d} = -Q_{b\ ea}^{\ d} + Q_{b\ ae}^{\ d}$. For second order derivatives we have the condition $W_{b\ c,de}^{\ a} - W_{b\ c,ed}^{\ a} = T_{d\ e}^{\ f} W_{b\ c,f}^{\ a}$ of integrability.

(III.11) Special Solutions Using Transformation Integrability.

Using the relation (III.10.1) for the integrability of $v_{(c)}^{i}$ we can find a family of special solutions in which the Riemann and Ricci tensors can be explicitly calculated. The $Y_{b\ c}^{\ a}$ are given in terms of $T_{b\ c}^{\ a}$ which allows easy transformation into coordinates, and there are no pointwise geometrical restrictions on $T_{b\ c}^{\ a}$ as (III.7.1) implied in the cases considered in (III.9). As in (III.9) we assume the geometry structure scalars are given as a linear combination of projections on their symmetry space of the contractions (III.8.1) of $T_{b\ c}^{\ a}$ with itself. The most general such expression is of the form

$$\begin{aligned} S_{ijk\ell} = {} & \alpha_1\left(F_{ijk\ell} + F_{ij\ell k} - F_{i\ell kj} - F_{\ell ikj}\right) \\ & + \alpha_2\left(F_{ik\ell j} + F_{ij\ell k} - F_{ki\ell j} - F_{\ell ikj}\right) + \alpha_3\left(F_{i\ell jk} - F_{jik\ell} + F_{\ell ijk} - F_{ji\ell k}\right) \\ & + \alpha_4\left(F_{ikj\ell} - F_{\ell ijk} + F_{ji\ell k} - F_{kij\ell}\right) + \beta_1\left(M_{ijk\ell} + M_{kj\ell i} + 2M_{\ell jki}\right) \\ & + \beta_2\left(M_{ik\ell j} + M_{kij\ell} + M_{\ell ijk} + M_{\ell kij}\right) + \beta_3\left(M_{i\ell jk} + M_{k\ell ij} + M_{\ell ijk} + M_{\ell kij}\right) \\ & + \beta_4\left(M_{jk\ell i} - 2M_{j\ell ik} + M_{jik\ell}\right) + \gamma\left(C_{ijk\ell} + 2C_{ikj\ell} + C_{i\ell jk}\right), \end{aligned}$$

where the 9 coefficients α_1, α_2, α_3, α_4, β_1, β_2, β_3, β_4, γ are arbitrary constants. Without explicitly assuming dimension 4 and using the volume element as a dual transformation, the only natural way to express $Y_{j\ k}^{\ i}$ in terms of $T_{j\ k}^{\ i}$ is by taking $Y_{j\ k}^{\ i} = (c-1)\left(T^{i}_{\ jk} + T^{i}_{\ kj}\right)$

which gives the appropriate symmetry. We assume c is a constant parameter. If $c = 0$ we see that $v^i_{(b),a} = \frac{1}{2}\left(T_b{}^d{}_a + Y_b{}^d{}_a\right)v^i_{(d)}$ becomes $v^i_{(b),a} = \left\{{}^{\,d}_{b\;a}\right\}v^i_{(d)}$ using (III.1.3) and so therefore $v^{(i)}_{b;a} = 0$ for each fixed i. Since this covector $\underset{\sim}{v}^{(i)}$ (for each i) is covariantly constant in the frame component system, it is also so in the homogeneous system, say $v^{(i)}_{j;k} = 0$. But $v^{(i)}_j = \delta^i_j$ here, so that $\left\{{}^{\,i}_{j\;k}\right\} = 0$ in the homogeneous system. Thus the space-time is flat and the Riemann tensor is zero, in the case $c = 0$.

Using the above forms for $S_{ijk\ell}$ and $Y_j{}^i{}_k$ we can substitute into the integrability condition (III.10.1). When differentiating $Y_j{}^i{}_k$, we differentiate the $T^i{}_{jk}$ and $T^i{}_{kj}$ terms, express in terms of $S_{ijk\ell}$ by lowering indices, and substitute for $S_{ijk\ell}$ above. If we do this, and collect up coefficients, assuming complete independence except for natural symmetries, we find that $\alpha_2 = \alpha_4 = \beta_1 = \beta_2 = 0$, $\alpha_1 = \dfrac{(c-1)^2}{2(4-3c)}$, $\alpha_3 = \dfrac{(c-1)(2c-3)}{2(4-3c)}$, $\beta_3 = \frac{1}{2}(1-c)$, $\beta_4 = \frac{1}{2}$, $\gamma = \dfrac{c-1}{2(4-3c)}$.

Therefore we have the geometry structure scalars given as

$$
\begin{aligned}
S_{ijk\ell} = \frac{(c-1)^2}{2(4-3c)}&\left(F_{ijk\ell} + F_{ij\ell k} - F_{i\ell kj} - F_{\ell ikj}\right)\\
&+ \frac{(c-1)(2c-3)}{2(4-3c)}\left(F_{i\ell jk} - F_{jik\ell} + F_{\ell ijk} - F_{ji\ell k}\right)\\
+ \frac{1}{2}(1-c)&\left(M_{i\ell jk} + M_{k\ell ij} + M_{\ell ijk} + M_{\ell kij}\right) + \frac{1}{2}\left(M_{jk\ell i} - 2M_{j\ell ik} + M_{jik\ell}\right)\\
&+ \frac{(c-1)}{2(4-3c)}\left(C_{ijk\ell} + 2C_{ikj\ell} + C_{i\ell jk}\right).
\end{aligned}
\tag{III.11.1}
$$

If we write out the reduced geometry structure scalars (III.8) as $\tilde{S}_{ijk\ell} = \tilde{\alpha}_1(F_{ijk\ell} + \cdots) + \tilde{\alpha}_2(F_{ik\ell j} + \cdots) + \cdots$ etc. exactly as for $S_{ijk\ell}$ with $\sim$ over all the parameters, we see that $\tilde{\gamma} = \gamma + \frac{1}{8}$, $\tilde{\alpha}_1 = \alpha_1 + \frac{1}{4}$, $\tilde{\beta}_1 = \beta_1 + \frac{1}{2}$ and the others are unchanged, i.e. $\tilde{\alpha}_2 = \alpha_2$, $\tilde{\alpha}_3 = \alpha_3$ etc. We write out the Riemann tensor $\mathcal{R}_{ijk\ell} = \frac{1}{2}\big(-\tilde{S}_{ki\ell j} + \tilde{S}_{kj\ell i} -$

$\tilde{S}_{ikj\ell} + \tilde{S}_{i\ell jk})$ using $\tilde{\alpha}_2 = \tilde{\alpha}_4 = \tilde{\beta}_2 = 0$ as

$$\mathcal{R}_{ijk\ell} = (\tilde{\alpha}_1 - \tilde{\alpha}_3)\left(F_{\ell ikj} + F_{\ell ijk} - F_{\ell jki} - F_{j\ell ki} - F_{ik\ell j} + F_{i\ell kj} - F_{ikj\ell} + F_{i\ell jk}\right)$$
$$+ \frac{\tilde{\beta}_1 - \tilde{\beta}_4}{2}\left(M_{\ell ikj} + M_{kij\ell} + 2M_{jik\ell} + M_{\ell jik} + M_{kj\ell i} + M_{j\ell ki} + M_{jki\ell} + M_{ik\ell j}\right.$$
$$\left. + 2M_{ij\ell k} + M_{i\ell jk} + 2M_{\ell kij} + 2M_{k\ell ji}\right)$$
$$+ 2\tilde{\gamma}\left(C_{\ell ikj} + 2C_{\ell kij} + C_{\ell jik}\right),$$

and using $\tilde{\beta}_1 = \tilde{\beta}_4 = \frac{1}{2}$, $\tilde{\beta}_3 = \frac{1}{2}(1-c)$, $\tilde{\gamma} = \frac{1}{8} + \frac{c-1}{2(4-3c)}$, $\tilde{\alpha}_1 = \frac{1}{4} + \frac{(c-1)^2}{2(4-3c)}$, $\tilde{\alpha}_3 = \frac{(c-1)(2c-3)}{2(4-3c)}$ we find that

$$\mathcal{R}_{ijk\ell} = \frac{c(3-2c)}{4(4-3c)}\left(F_{\ell ikj} + F_{\ell ijk} - F_{\ell jki} - F_{j\ell ki} - F_{ik\ell j} + F_{i\ell kj} - F_{ikj\ell} + F_{i\ell jk}\right)$$
$$+ \frac{c}{4(4-3c)}\left(C_{\ell ikj} + 2C_{\ell kij} + C_{\ell jik}\right). \qquad \text{(III.11.2)}$$

This is the final form for our Riemann Tensor. Thus we see by a different method altogether that the space is flat for $c = 0$. If we work out the Ricci Tensor $\mathcal{R}_{jk} = \eta^{i\ell}\mathcal{R}_{ijk\ell}$ using (III.8.3) we find that

$$\mathcal{R}_{jk} = \frac{c(3-2c)}{4(4-3c)}\left(4A_{(kj)} - 2D_{(kj)} - E_{kj}\right) + \frac{2c^2}{4(4-3c)} B_{kj}. \qquad \text{(III.11.3)}$$

There are several special cases which it is of interest to examine. If we take $c = \frac{3}{2}$ then $\mathcal{R}_{jk} = -\frac{9}{4} B_{jk}$ and the differential equation for $T_j{}^i{}_k$ is

$$T_{kji,\ell} = \frac{1}{2}\left(M_{jk\ell i} + M_{jik\ell}\right) - \frac{1}{4}\left(F_{ijk\ell} + F_{ij\ell k} - F_{i\ell kj} - F_{\ell ikj}\right)$$
$$- \frac{1}{4}\left(M_{i\ell jk} + M_{k\ell ij} + M_{\ell ijk} + M_{\ell kij}\right) - \frac{1}{2}\left(C_{ijk\ell} + 2C_{ikj\ell} + C_{i\ell jk}\right).$$

(III.11.4)

On the other hand if $c = 1$, then we obtain $T_{kji,\ell} = \frac{1}{2}\left(M_{jk\ell i} + M_{jik\ell}\right)$ and $R_{jk} = \frac{1}{2} B_{jk} - \frac{1}{4} E_{jk} + A_{(jk)} - \frac{1}{2} D_{(jk)}$. It is easy to check directly for this differential equation that $T_{j\ k,\ell m}^{\ i} - T_{j\ k,m\ell}^{\ i} = T_{\ell\ m}^{\ a} T_{j\ k,a}^{\ i}$ holds identically without any restrictions on $T_{j\ k}^{\ i}$ such as we found in (III.9). This means that the equivalent form (III.7.1) also holds. The verification of this result for (III.11.1) with $c \neq 1$ is extremely tedious, because substitution gives numerous cubic terms in T with 2 contractions each.

It would be nice to find $T_{j\ k}^{\ i}$ that diagonalizes R_{jk} in (III.11.3) in the orthogonal part (1,2,3 components) and such that the condition (III.11.1) on the geometry structure scalars would maintain $T_{j\ k}^{\ i}$ in that form. This would give us a solution in adapted frame components, and the restriction on $\left(T_{j\ k}^{\ i}\right) \in \mathbb{R}^{24}$ gives us a *constraint manifold* if these conditions are satisfied.

(III.12) The Omnidirectional Solutions.

We are interested here in imposing conditions of spatial isotropy on the Ricci coefficients directly, in order to limit the class of space-times to those which can be handled with some degree of simplicity. The natural condition we obtain, which is especially useful in cosmology, is that of omnidirectionality. A space-time is *omnidirectional* if no particular spatial direction orthogonal to the matter flow can be singled out by any process whatsoever (including frame component differentiation) from the Ricci rotation coefficients of some adapted frame component system. This definition includes a dependence on the matter flow of the material medium but is independent of any homogeneous coordinate systems. Aside from being simple, these solutions are also physically interesting and reasonable. Of necessity, any solution using

frame components in which no spatial direction is distinguished must have equal principal stresses and no heat flow, and has the energy tensor type of a "perfect fluid". In 4 dimensions we then have $T_a{}^c{}_b = f\gamma_{[a}{}^c u_{b]} + \hat{f}\epsilon_a{}^c{}_b$, and the $\hat{f}$ term is missing in $n+1$ dimensions generally. Here $\gamma_{AB} = \delta_{AB}$, $\gamma_{4a} = \gamma_{a4} = 0$, $u_4 = -1$, $u_A = 0$, $\epsilon_{abc} = \epsilon_{abcd}u^d$, $A,B = 1,2,3$, $a,b = 1,2,3,4$ and η_{jk} is used to raise and lower indices. The omnidirectional condition implies $f_{,\ell} = \dot{f}u_\ell$ and $\hat{f}_{,\ell} = \dot{\hat{f}}u_\ell$ so the motion is irrotational and geodesic with local time coordinate τ upon which f and $\hat{f}$ depend. If we wish a transformation into coordinates we may write

$$\gamma_a{}^c{}_b = g\gamma_{(a}{}^c u_{b)} + hu^c\gamma_{ab} + ku^c u_a u_b$$

where $g_{,\ell} = \dot{g}u_\ell$, $h_{,\ell} = \dot{h}u_\ell$, $k_{,\ell} = \dot{k}u_\ell$ so that g, h and k are also functions of τ. The functions $f, \hat{f}, g, h, k$ are clearly integrable, i.e. $f_{,\ell m} - f_{,m\ell} = 0 = T_\ell{}^k{}_m f_{,k}$ etc., and the Jacobi identity implies $\dot{\hat{f}} = \frac{1}{2}f\hat{f}$. If we substitute into the integrability condition (III.10.1) for $v^i_{(c)}$ we get

$$\begin{aligned}
\dot{\hat{f}} &= \frac{1}{2}f\hat{f} && (1)\\
\frac{1}{2}\dot{f} &= \frac{1}{8}f^2 + \frac{1}{2}\dot{g} + \frac{1}{8}g^2 + \frac{1}{4}kg - \frac{1}{4}f(k+g) && (2)\\
0 &= \frac{1}{2}\hat{f}^2 + \frac{1}{4}h(f-g) && (3)\\
0 &= \dot{h} - \frac{1}{2}hk - \frac{3}{4}hf - \frac{1}{4}hg\,. && (4)
\end{aligned} \qquad \text{(III.12.1)}$$

If we differentiate (3) and use (1), (2) and (4) to eliminate all derivatives in the resulting equation we obtain an identity. Thus it is easy to see that if f and $\hat{f}$ are given as functions of τ subject to the Jacobi identity $\dot{\hat{f}} = \frac{1}{2}f\hat{f}$ we can find functions g, h, k of τ satisfying (1) through (4) in more than one way. In the particu-

lar case $\hat{f} = 0$ we may take $g = f$, $h = k = 0$ as a particularly simple form that gives us

$$v^{i}_{(a),b} = \frac{1}{2}\left(T_{a\ b}^{\ c} + Y_{a\ b}^{\ c}\right)v^{i}_{(c)} = \frac{f}{2}\,\gamma_{a}^{c}u_{b}v^{i}_{(c)}\ . \qquad \text{(III.12.2)}$$

If $\hat{f} \neq 0$ and we are given functions $f, \hat{f}$ of τ satisfying $\dot{\hat{f}} = \frac{1}{2} f\hat{f}$ we can obtain g, h, k satisfying (III.12.1) in the following manner. Solve the differential equation $\dot{g} - \dot{f} = \hat{f}^2 + \frac{1}{4}(g^2 - f^2)$ for g with any initial conditions, then put $h = \frac{2\hat{f}^2}{g-f}$ and $k = -g-h$. We obtain $Y_{a\ b}^{\ c} = g\delta^{c}_{(a}u_{b)} + hu^{c}\eta_{ab}$ from which we can integrate to obtain homogeneous coordinates. This is not the only solution of (III.12.1) for $\hat{f} \neq 0$, but includes all cases for which $k + g + h = 0$.

The Ricci tensor can be obtained from (III.4.5) and substituting we obtain

$$R_{jk} = \left(\frac{\dot{f}}{2} - \frac{\hat{f}^2}{2} - \frac{3}{4}f^2\right)\gamma_{jk} + \left(\frac{3}{4}f^2 - \frac{3}{2}\dot{f}\right)u_j u_k$$

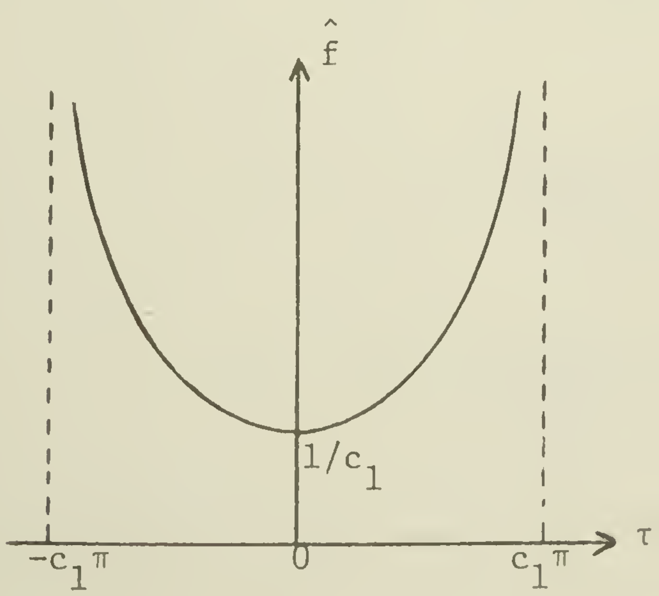
so that putting $P = -\sigma_1 = -\sigma_2 = -\sigma_3$ as the pressure we have that $P = \frac{1}{4\kappa}(4\dot{f} - 3f^2 - \hat{f}^2)$ and $\rho c^2 + \epsilon = \frac{3}{4\kappa}(\hat{f}^2 + f^2)$ using Einstein's equations, where f and $\hat{f}$ are arbitrary functions of τ subject only to the condition $\dot{\hat{f}} = \frac{1}{2} f\hat{f}$.

(III.13) <u>The Zero Pressure Case</u>.

The dust solution obtained by putting $P = 0$ is of great interest cosmologically. We obtain two first order differential equations $4\dot{f} = 3f^2 + \hat{f}^2$ and $\dot{\hat{f}} = \frac{1}{2} f\hat{f}$. These can be combined to form a single equation $8\hat{f}\ddot{\hat{f}} = 20\hat{f}^2 + \hat{f}^4$ for $\hat{f}$ or $2\ddot{f} = 5f\dot{f} - \frac{3}{2}f^3$ for f. In the case $\hat{f} = 0$ we can solve explicitly for f to get $f = \frac{4}{3\tau + c_o}$ for a constant c_o of integration. For $3\tau > -c_o$ and τ positive, $f \to 0$ as $\tau \to +\infty$ so for large τ we have f^2 decreasing so

$\rho c^2 + \epsilon$ is decreasing and we have an expanding universe that continues always to expand. We call this the *parabolic solution* since it corresponds in Newtonian mechanics to the matter having just barely enough energy to escape to infinity after expanding from the initial infinite density singularity at $\tau = -\frac{c_0}{3}$.

Now let us examine the $\hat{f} \neq 0$ case (which we have only for $n + 1 = 4$). We shall see this gives us the *elliptical solutions* in which, after expanding from the infinite density singularity, it has insufficient energy and contracts back. Of course, for these dust solutions, all the energy in the rest frame is of the form of mass and no heat is stored by the system so we may put $\epsilon = 0$. It is easy to see that $\dot{\rho} = \frac{3}{2} f\rho$ by differentiating $\hat{f}^2 + f^2$. Dividing this equation by $\dot{\hat{f}} = \frac{1}{2} f\hat{f}$ and integrating we have $\rho = c_0\hat{f}^3$ for some constant c_0 in the case where $\hat{f} \neq 0$. Hence $\hat{f}$ must remain non-zero, and without loss of generality we can insist that it is positive. For some c_1 we have $\hat{f}^2 + f^2 = c_1\hat{f}^3$ so $f = \pm\hat{f}\sqrt{c_1\hat{f} - 1}$ and $\mp\tau = 2\int \frac{df}{\hat{f}^2\sqrt{c_1\hat{f} - 1}} + c_2$. This can be found in a table of integrals, and ignoring c_2, the time shift, we have $\mp\frac{\tau}{2} = \frac{\sqrt{c_1\hat{f} - 1}}{\hat{f}} + c_1 \tan^{-1}\left(\sqrt{c_1\hat{f} - 1}\right)$. We require $\hat{f} \geq \frac{1}{c_1} > 0$ at all times. If we graph the curve of this function we find that this universe has a finite lifetime. Expansion



occurs from $-c_1\pi < \tau < 0$ and it collapses for $0 < \tau < c_1\pi$. The solution for τ too close to $-c_1\pi$ and $+c_1\pi$ will not be valid because high density will destroy the $P = 0$ approximation. Our mass density is given at each τ by

$\rho = \dfrac{3c_1\hat{f}^3}{4\kappa c^2}$. Notice that $\dot{\hat{f}} = -\dfrac{d\hat{f}}{d\tau}$ and $\tau_{,\ell} = -u_\ell$ for proper time sense, i.e. $\dot{\tau} = -1$.

Finally we have the *hyperbolic solutions* in which the universe expands to infinity with energy to spare. These solutions we will obtain in (III.19) where we discuss specific unidirectional cases. They are physically but not geometrically omnidirectional, i.e. the Ricci coefficients single out a spatial direction which cannot be detected through the physical parameters. It is worth noting that in these dust solutions we have introduced no cosmological parameter Λ and in fact $\Lambda = 0$.

(III.14) Integration for the Metric.

As an example to find a form for the metric tensor in (homogeneous) coordinates, we take the omnidirectional solution with $\hat{f} = 0$, $g = f$, $h = k = 0$. From (III.12.2) we obtain $v^i_{(a),b} = \frac{f}{2}\gamma^c_a u_b v^i_{(c)}$. Then $ds^2 = g_{ij}dx^i dx^j$ where $g^{ij} = v^i_{(a)}v^j_{(b)}\eta^{ab}$ and $v^i_{(a),b} = v^i_{(a)\natural j}v^j_{(b)}$ where $\natural$ is the partial derivative in homogeneous coordinates. Then $v^i_{(a),1} = v^i_{(a),2} = v^i_{(a),3} = 0$ and $v^i_{(a),4} = -\frac{f}{2}\gamma^c_a v^i_{(c)}$, so $v^i_{(4),4} = 0$ and $v^i_{(J),4} = -\frac{f}{2}v^i_{(J)}$ so $\dfrac{v^i_{(J),4}}{v^i_{(J)}} = -\dfrac{f}{2}$ and $\ln(v^i_{(J)}) = -\frac{1}{2}\int_0^\tau f(\sigma)d\sigma +$ constant. Hence if we put $v^i_{(c)} = \delta^i_c$ at $\tau = 0$ we have $v^i_{(J)} = \delta^i_J \exp\left(-\frac{1}{2}\int_0^\tau f(\sigma)d\sigma\right)$, $J = 1,2,3$, and $v^i_{(4)} = \delta^i_4$. From this we obtain the metric components g_{ij} as $g_{11} = g_{22} = g_{33} = \exp\left(\int_0^\tau f(\sigma)d\sigma\right)$, $g_{44} = -1$ with the off diagonal components zero. Putting $x^1 = x$, $x^2 = y$, $x^3 = z$, $x^4 = \tau$ we have the metric

$$ds^2 = -d\tau^2 + \exp\left(\int_0^\tau f(\sigma)d\sigma\right)(dx^2 + dy^2 + dz^2).$$

We recognize this as the Robertson-Walker metric with the constant

$k = 0$, if we put $dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2$ in place of $dx^2 + dy^2 + dz^2$.

Normally, integration for the metric is a complicated and tiresome procedure that is worth avoiding, particularly when all the physical information is available from the Ricci rotation coefficients $T_j{}^i{}_k$. If we work out the Christoffel symbols in these homogeneous coordinates we find that $\left\{{}_{I\ I}^{\ 4}\right\} = \frac{f(\tau)}{2} \exp\left(\int_0^\tau f(\sigma)d\sigma\right)$ and $\left\{{}_{4\ I}^{\ I}\right\} = \frac{f(\tau)}{2}$ where $I = 1,2,3$ for each I with no sum on I. All others are zero except for symmetry, i.e. $\left\{{}_{I\ 4}^{\ I}\right\}$. We can evaluate

$$R_{jk} = \left\{{}_{j\ i}^{\ i}\right\}_{,k} - \left\{{}_{j\ k}^{\ i}\right\}_{,i} + \left\{{}_{a\ k}^{\ i}\right\}\left\{{}_{j\ i}^{\ a}\right\} - \left\{{}_{a\ i}^{\ i}\right\}\left\{{}_{j\ k}^{\ a}\right\}$$

in these coordinates and we get (using $\dot{f} = -f_{,4}$ since $f_{,\ell} = \dot{f} u_\ell$)
$R_{jk} = \left(\frac{3}{4} f^2 - \frac{3\dot{f}}{2}\right) u_j u_k + \left(\frac{\dot{f}}{2} - \frac{3}{4} f^2\right)\gamma_{jk}$ exactly as before (with $\hat{f} = 0$)
where $\gamma_{4i} = 0$ and $\gamma_{ij} = 0$, $i \neq j$ with $\gamma_{II} = \exp\left(\int_0^\tau f(\sigma)d\sigma\right)$, $I = 1,2,3$, and $u_4 = -1$, $u_1 = u_2 = u_3 = 0$. The tensor transformation through $v^i_{(a)}$ back to frame components, gives us the usual form for γ_{ab} and u_a.

(III.15) Unidirectional Space-Times.

We look for a form for the Ricci rotation coefficients which singles out just one spatial direction orthogonal to the flow. This direction can be taken as r^a, the first principal stress direction, since the unit normal direction must have zero derivative — otherwise another direction would be singled out. This would apply for instance in spherical symmetry.

In adapted frame components, $\epsilon_{jk\ell} = 6r_{[j}s_k t_{\ell]}$ and $\epsilon_{ijk\ell} = -24r_{[i}s_j t_k u_{\ell]}$ so $\epsilon_{1234} = 1$. We put $\Lambda_{jk} = \epsilon_{jk\ell} r^\ell = s_j t_k - s_k t_j$ and $\bar{\epsilon}_{ijk} = \epsilon_{ijk\ell} r^\ell$. The most general form for $T_j{}^i{}_k$ subject to the

unidirectional symmetry is $T_j{}^i{}_k = f\gamma^i{}_{[j}u_{k]} + \hat{f}\epsilon_j{}^i{}_k + gu^i\Lambda_{jk} + hr^i\Lambda_{jk} + k\Lambda^i{}_{[j}u_{k]} + p\bar{\epsilon}_j{}^i{}_k + q\Lambda^i{}_{[j}r_{k]} + r\gamma^i{}_{[j}r_{k]} + su^ir_{[j}u_{k]} + tr^ir_{[j}u_{k]}$, where $f_{,\ell} = \dot{f}u_\ell + f'r_\ell$, $\hat{f}_{,\ell} = \dot{\hat{f}}u_\ell + \hat{f}'r_\ell$, $g_{,\ell} = \dot{g}u_\ell + g'r_\ell$ and similarly for h, k, p etc. The functions r, s, t are not to be confused with the vectors $\underset{\sim}{r}$, $\underset{\sim}{s}$, $\underset{\sim}{t}$ which have unit length and components $r^i = \delta^i_1$, $s_j = \delta^2_j$ etc. We see that $f_{,\ell m} = \ddot{f}u_\ell u_m + f''r_\ell r_m + \dot{f}'u_\ell r_m + f'^{\cdot} r_\ell u_m$, and so $f_{,\ell m} - f_{,m\ell} = (\dot{f}' - f'^{\cdot})(u_\ell r_m - u_m r_\ell) = T_\ell{}^c{}_m f_{,c}$. This must hold similarly for the other functions, and so if $f_{,c}$, $g_{,c}$, $\hat{f}_{,c}$ etc. are not all parallel to the same vector we require that $g = p$ and $\hat{f} = h$. Under these circumstances, the motion is irrotational, i.e. $\omega_{ij} = 0$ since $T_A{}^4{}_B = 0$.

In general we can write out the integrability conditions as

$$
\begin{aligned}
\dot{f}' - f'^{\cdot} &= \tfrac{1}{2}\left(\dot{f}s - f'(f+t)\right), & 0 &= \dot{f}(p-g) + f'(h-\hat{f}) \\
\dot{\hat{f}}' - \hat{f}'^{\cdot} &= \tfrac{1}{2}\left(\dot{\hat{f}}s - \hat{f}'(f+t)\right), & 0 &= \dot{\hat{f}}(p-g) + \hat{f}'(h-\hat{f}) \\
\dot{g}' - g'^{\cdot} &= \tfrac{1}{2}\left(\dot{g}s - g'(f+t)\right), & 0 &= \dot{g}(p-g) + g'(h-\hat{f}) \\
\dot{h}' - h'^{\cdot} &= \tfrac{1}{2}\left(\dot{h}s - h'(f+t)\right), & 0 &= \dot{h}(p-g) + h'(h-\hat{f})
\end{aligned}
\qquad \text{(III.15.1)}
$$

and similarly for k, p, q, r, s, t.

A space-time (not necessarily unidirectional) is said to be *flow-static* if the Ricci coefficients for a set of adapted frame components satisfy $T_j{}^i{}_{k,4} = 0$ and $T_J{}^I{}_4 = 0$ for all $i,j,k = 1,\cdots,4$, $I,J = 1,2,3$. This of course is defined locally, meaning it is flow-static on a neighborhood U or perhaps on all of M. This means all 4-derivatives of $T_j{}^i{}_k$, say $T_j{}^i{}_{k,\ell 4}$, $T_j{}^i{}_{k,\ell m4}$ etc. vanish. For a unidirectional space time, this means all 4-derivatives of f, $\hat{f}$, g, h etc. vanish. A space-time is said to be *flow-stationary* if $\theta_{ab} = 0$, i.e. the

motion is rigid. Since for a scalar field ϕ, $L_{\phi \underset{\sim}{u}} g_{ab} = 0$ if and only if $\theta_{ab} = 0$ and $\dot{u}_b = (\ln \phi)_{,b}$, a flow-stationary space-time with a gravitational potential is stationary. Since $T_J{}^I{}_4 = 0 \Rightarrow \theta_{AB} = 0$, a flow-static space is always flow stationary (see (III.9.4)). For a relationship of flow-static to the usual definition of stationary see (V.18).

For a unidirectional space-time an interesting special case occurs when there exists scalars τ and x defined locally with $\tau_{,\ell} = -u_\ell$ and $x_{,\ell} = r_\ell$. This holds if and only if $T_i{}^1{}_j$ and $T_i{}^4{}_j$ are zero, i.e. $g = p$, $s = 0$, $f + t = 0$, $h = \hat{f}$. In this case the derivatives commute, i.e. $\dot{f}' = f'^{\cdot}$, $\dot{g}' = g'^{\cdot}$, $\dot{h}' = h'^{\cdot}$ etc. and we have two naturally defined coordinates τ and x forming part of a coordinate system, these scalar functions being adapted to the metric.

We obtain the kinematic parameters in the unidirectional case from (III.9.4) as $\dot{u}_a = -\frac{s}{2} r_a$, $\omega_{AB} = \left(\frac{g-p}{2}\right)\Lambda_{AB}$,

$$\theta_{AB} = \frac{f}{2}\gamma_{AB} + \frac{t}{2} r_A r_B , \quad \theta = \frac{3f}{2} + \frac{t}{2} .$$

(III.16) <u>The Jacobi Identity</u>.

The coordinate torsion $T_j{}^i{}_k$ for a unidirectional space must satisfy the Jacobi identity $T_{[j}{}^i{}_{k,\ell]} + T_a{}^i{}_{[\ell} T_j{}^a{}_{k]} = 0$. If we work out $T_j{}^i{}_{k,\ell}\epsilon^{jk\ell m}$ by differentiating the formula for $T_j{}^i{}_k$ in (III.15) we find that

$$T_j{}^i{}_{k,\ell}\epsilon^{jk\ell m} = (f'-\dot{r})\Lambda^{im} + (k' - \dot{q} + 2\dot{h} - 2p')r^i r^m +$$
$$(\dot{q} - k' + 2p' - 2\dot{\hat{f}})\gamma^{im} + 2(h'-\hat{f}')r^i u^m + 2(\dot{g}-\dot{p})u^i r^m + 2(g'-p')u^i u^m .$$

In calculating this we should recall that in adapted frame components $\varepsilon^{ijk\ell} = \epsilon^{ijk\ell}$, $\varepsilon_{ijk\ell} = -\epsilon_{ijk\ell}$, $\varepsilon_{ijk} = -\epsilon_{ijk}$, $\varepsilon^{ijk} = -\epsilon^{ijk}$

where $\underset{\sim}{\epsilon}$ is the tensor whose indices are raised and lowered using the metric η_{ij}. Furthermore $\epsilon_{ijk\ell} = -24r_{[i}s_j t_k u_{\ell]}$, $\epsilon^{ijk\ell} = 24r^{[i}s^j t^k u^{\ell]}$, $\epsilon_{1234} = \epsilon^{1234} = 1$, $\epsilon_{123} = \epsilon^{123} = 1$, $\epsilon_{ijk} = 6r_{[i}s_j t_{k]}$, $\epsilon^{ijk} = 6r^{[i}s^j t^{k]}$, $\bar{\epsilon}_{ijk} = 6u_{[i}s_j t_{k]}$, $\bar{\epsilon}_{423} = -1$, $\epsilon_{ijk}\epsilon^{i\ell m} = 2\gamma^{\ell}_{[j}\gamma^{m}_{k]} = \gamma^{\ell}_{j}\gamma^{m}_{k} - \gamma^{\ell}_{k}\gamma^{m}_{j}$, $\epsilon^{ijn}\epsilon_{ijk} = 2\gamma^{n}_{k}$, $\epsilon_{ijk\ell}\epsilon^{imnp} = 6\delta^{m}_{[j}\delta^{n}_{k}\delta^{p}_{\ell]}$, $\epsilon^{ijk\ell}\epsilon_{ijmn} = 4\delta^{k}_{[m}\delta^{\ell}_{n]}$, $\epsilon^{ijk\ell}\epsilon_{ijkn} = 6\delta^{\ell}_{n}$, $\epsilon_{ijk} = \epsilon_{ijk\ell}u^{\ell}$, $\epsilon^{ijk} = -\epsilon^{ijk\ell}u_{\ell}$.

Next we evaluate $T_a{}^i{}_\ell T_j{}^a{}_k \epsilon^{jk\ell m}$ and find that

$$T_a{}^i{}_\ell T_j{}^a{}_k \epsilon^{jk\ell m} = \Lambda^{im}\left(\frac{rf}{2} + \frac{tr}{2} - \frac{fs}{2}\right) + r^i r^m\left(-fh - 2\hat{f}t - \frac{ks}{2} + ps + \frac{qf}{2} + \frac{qt}{2} + th\right) + \gamma^{im}\left(f\hat{f} + \hat{f}t + \frac{ks}{2} - ps - \frac{qf}{2} - \frac{qt}{2}\right)$$

$$+ r^i u^m\left(fg - fp - 2rh + 2\hat{f}r + tg - tp\right) + u^i r^m(-2fg + 2fp - s\hat{f} + sh)$$

$$+ u^i u^m(-2gr + 2pr + sg - sp).$$

Therefore the six conditions implied by the Jacobi identity are

$$f' - \dot{r} + \frac{rf}{2} + \frac{tr}{2} - \frac{fs}{2} = 0, \tag{1}$$

$$k' - \dot{q} + 2\dot{h} - 2p' - fh - 2\hat{f}t - \frac{ks}{2} + ps + \frac{qf}{2} + \frac{qt}{2} + th = 0, \tag{2}$$

$$\dot{q} - k' + 2p' - 2\dot{\hat{f}} + f\hat{f} + \hat{f}t + \frac{ks}{2} - ps - \frac{qf}{2} - \frac{qt}{2} = 0, \tag{3}$$

$$2h' - 2\hat{f}' + fg - fp - 2rh + 2\hat{f}r + tg - tp = 0, \tag{4}$$

$$2\dot{g} - 2\dot{p} - 2fg + 2fp - s\hat{f} + sh = 0, \tag{5}$$

$$2g' - 2p' - 2gr + 2pr + sg - sp = 0. \tag{6}$$

We can add (2) and (3) to obtain the auxiliary result (7) which is given below and is simpler, namely

$$2\dot{h} - 2\dot{\hat{f}} + f\hat{f} - fh - \hat{f}t + th = 0. \tag{7}$$

We can use (7) to replace one of the more complicated equations (2) or

(3). These forms for the unidirectional Jacobi identity will be referred to as (III.16.1) through (III.16.7).

(III.17) The Ricci Tensor.

Next, we work out the Ricci tensor for the unidirectional space-time using the formula (III.4.5), namely

$$\mathcal{R}_{jk} = T_{(k,j)} + T^{i}{}_{(kj),i} + \frac{1}{2} B_{jk} - \frac{1}{4} E_{jk} + A_{(jk)} + \frac{1}{2} G_{jk}.$$

We can see from symmetry arguments that it contains function multiples of γ_{jk}, $r_j r_k$, $u_j u_k$, $r_{(j}u_{k)}$ only, and these coefficient functions are quadratic in the given functions (in $T_j{}^i{}_k$) and linear in their first derivatives. We can check that $T_j = -\frac{3f+t}{2} u_j - \frac{2r+s}{2} r_j$ and so

$$T_{(j,k)} = -\left(\frac{3\dot{f}+\dot{t}}{2}\right) u_j u_k - \left(\frac{2r'+s'}{2}\right) r_j r_k - \left(\frac{3f'+t'+2\dot{r}+\dot{s}}{2}\right) u_{(j}r_{k)},$$

and similarly we can evaluate $T^{i}{}_{(kj),i}$, $\frac{1}{2} B_{jk} = \frac{1}{2} T^{i}{}_{aj} T^{a}{}_{ik}$ and all the others. We find, after this lengthy calculation that the $\mathcal{R}_{jk}$ we obtain (which is automatically diagonal in the orthogonal part and therefore of the form for an adapted frame component system) is given by

$$\mathcal{R}_{jk} = u_j u_k \left[\frac{s'}{2} - \frac{3\dot{f}}{2} - \frac{\dot{t}}{2} + \frac{ft}{2} + \frac{3f^2}{4} + \frac{t^2}{4} - \frac{s^2}{4} - \frac{g^2}{2} + pg - \frac{p^2}{2} - \frac{rs}{2}\right]$$

$$+ \gamma_{jk}\left[\frac{\dot{f}}{2} - \frac{r'}{2} - \frac{g^2}{2} + \frac{h^2}{2} - \frac{\hat{f}^2}{2} + \frac{\hat{f}q}{2} - \frac{kp}{2} + \frac{p^2}{2} + \frac{r^2}{2} + \frac{rs}{4} - \frac{3f^2}{4}\right.$$

$$\left. - \frac{ft}{4} + \frac{gk}{2} - \frac{qh}{2}\right] + r_j r_k\left[-\frac{r'}{2} - \frac{s'}{2} + \frac{\dot{t}}{2} + \frac{g^2}{2} - h^2 - \frac{q\hat{f}}{2} + \frac{s^2}{4} + h\hat{f}\right.$$

$$\left. + \frac{kp}{2} - \frac{p^2}{2} - \frac{rs}{4} - \frac{gk}{2} + \frac{qh}{2} - \frac{3ft}{4} - \frac{t^2}{4}\right] + (r_k u_j + u_k r_j)\left[-\frac{f'}{2} - \frac{\dot{r}}{2} +\right.$$

$$\left. + \frac{g\hat{f}}{2} - \frac{\hat{f}p}{2} + \frac{hp}{2} - \frac{gh}{2} - \frac{fs}{4} + \frac{fr}{4} - \frac{tr}{4}\right].$$

Just for reference we record the intermediate values in the calculation.

$$T^{i}{}_{(kj),i} = \frac{t' + f' + \dot{s}}{2} r_{(k}u_{j)} + \frac{\dot{f} - r'}{2} \gamma_{jk} + \frac{\dot{t} + r'}{2} r_j r_k + \frac{s'}{2} u_j u_k ,$$

$$\begin{aligned} B_{jk} = {} & u_j u_k\left(-2kp + \frac{ft}{2} + \frac{3f^2}{4} + \frac{k^2}{2} + 2p^2 - \frac{s^2}{4} + \frac{t^2}{4}\right) + \gamma_{jk}\Big(2\hat{f}^2 - 2\hat{f}h - \hat{f}q - g^2 \\ & + 2gp + h^2 - \frac{k^2}{4} + kp - 2p^2 + \frac{q^2}{4} + \frac{r^2}{4} - \frac{f^2}{4}\Big) + r_j r_k\Big(-\frac{ft}{2} + 2\hat{f}h + g^2 - 2gp \\ & - h^2 + \frac{k^2}{4} - kp + 2p^2 - q\hat{f} + \frac{q^2}{4} + \frac{r^2}{4} + \frac{s^2}{4} - \frac{t^2}{4}\Big) \\ & + (r_k u_j + u_k r_j)\left(\frac{fr}{2} - \hat{f}k + 2\hat{f}p + \frac{kq}{2} - pq\right) , \end{aligned}$$

$$\begin{aligned} E_{jk} = {} & u_j u_k\left(2g^2 - 4pg + 2p^2 - \frac{s^2}{2}\right) + \gamma_{jk}\Big(-\frac{f^2}{2} + 2\hat{f}^2 - 2\hat{f}q + 2kp \\ & - 2p^2 + \frac{q^2}{2} + \frac{r^2}{2} - \frac{k^2}{2}\Big) + r_j r_k\Big(-tf - 4h\hat{f} + 2\hat{f}q + 2h^2 - 2kp + \frac{k^2}{2} \\ & + 2p^2 - \frac{q^2}{2} - \frac{r^2}{2} - \frac{t^2}{2}\Big) + (r_k u_j + u_k r_j)\left(-\frac{fs}{2} - 2g\hat{f} + 2\hat{f}p - 2hp + 2gh - \frac{st}{2}\right) , \end{aligned}$$

$$\begin{aligned} G_{jk} = {} & u_j u_k\left(\frac{3f^2}{4} + \frac{ft}{2} - \frac{k^2}{2} + 2kp - 2p^2 + \frac{t^2}{4}\right) + \gamma_{jk}\Big(-2\hat{f}^2 + 2\hat{f}h \\ & + \hat{f}q + gk - 2gp - pk + 2p^2 - qh\Big) + r_j r_k\Big(-2\hat{f}h + \hat{f}q - gk + 2gp \\ & + pk - 2p^2 - \frac{q^2}{2} + qh + \frac{r^2}{2} + \frac{s^2}{4}\Big) + (r_k u_j + u_j r_k)\Big(\frac{fr}{2} + \frac{fs}{4} \\ & + \frac{st}{4} + pq - \frac{kq}{2} + \hat{f}k - 2\hat{f}p\Big) , \end{aligned}$$

$$\begin{aligned} A_{(jk)} = {} & u_j u_k\left(-\frac{rs}{2} - \frac{s^2}{4}\right) + \gamma_{jk}\left(\frac{r^2}{2} + \frac{rs}{4} - \frac{3f^2}{4} - \frac{ft}{4}\right) + r_j r_k\Big(-\frac{r^2}{2} \\ & - \frac{rs}{4} - \frac{3ft}{4} - \frac{t^2}{4}\Big) + (r_k u_j + u_k r_j)\left(-\frac{rf}{4} - \frac{fs}{2} - \frac{tr}{4} - \frac{ts}{4}\right) . \end{aligned}$$

Substituting the expression for R_{jk} into (III.9.2) we obtain the principal stresses, mass and energy density and heat flow from

Einstein's equations as

$$\sigma_2 = \sigma_3 = -\frac{1}{2\kappa}\left(2\dot{f} - r' - s' + \dot{t} - \frac{h^2}{2} - \frac{\hat{f}^2}{2} - \frac{3f^2}{2} + \frac{r^2}{2} - \frac{3ft}{2} + \frac{s^2}{2} + h\hat{f} - \frac{t^2}{2} + \frac{g^2}{2} - pg + \frac{p^2}{2} + \frac{rs}{2}\right),$$

$$\sigma_1 = -\frac{1}{2\kappa}\left(2\dot{f} - \frac{g^2}{2} + \frac{3h^2}{2} - \frac{\hat{f}^2}{2} + \hat{f}q - kp + \frac{3p^2}{2} + \frac{r^2}{2} + rs - \frac{3f^2}{2} + gk - qh - pg - h\hat{f}\right),$$

$$\rho c^2 + \epsilon = -\frac{1}{2\kappa}\left(-2r' + \frac{h^2}{2} - \frac{3\hat{f}^2}{2} - \frac{3f^2}{2} + \frac{3r^2}{2} - ft + h\hat{f} - \frac{3g^2}{2} + \hat{f}q - kp + \frac{p^2}{2} + gk - qh + pg\right),$$

$\alpha^1 = 1, \quad \alpha^2 = \alpha^3 = 0,$ and

$$\lambda = \frac{1}{\kappa}\left(\frac{f'}{2} + \frac{\dot{r}}{2} - \frac{g\hat{f}}{2} + \frac{\hat{f}p}{2} - \frac{hp}{2} + \frac{gh}{2} + \frac{fs}{4} - \frac{fr}{4} + \frac{tr}{4}\right).$$

We have formulated the Einstein equations in complete generality for the unidirectional space-times. There are great simplifications in special cases, for instance if $h = \hat{f}$ and $g = p$ the Jacobi Identities (III.16.4) through (7) are trivial and we need only consider (1) and (2) explicitly. Likewise the Ricci tensor (or equivalently σ_I, $\rho c^2 + \epsilon$, λ above) and the integrability conditions (III.15.1) are simplified considerably, and many of the functions need not be considered.

(III.18) Examples of Unidirectional Space-Times.

We remarked earlier that the conditions $p = g$ and $\hat{f} = h$ were required if not all derivatives $f_{,a}$, $\hat{f}_{,a}$, $g_{,a}$, $h_{,a}$, $k_{,a}$ etc. are going to be parallel. With the notable exception of flow-static cases,

these will in general be non-parallel so that this condition $p = g$, $\hat{f} = h$ which greatly simplifies the Jacobi identity, integrability conditions and Ricci tensor should be considered. In this case

$$\sigma_2 = \sigma_3 = -\frac{1}{2\kappa}\left(2\dot{f} - r' - s' + \dot{t} - \frac{3f^2}{2} + \frac{r^2}{2} - \frac{3ft}{2} + \frac{s^2}{2} - \frac{t^2}{2} + \frac{rs}{2}\right),$$

$$\sigma_1 = -\frac{1}{2\kappa}\left(2\dot{f} + \frac{r^2}{2} + rs - \frac{3f^2}{2}\right), \quad \rho c^2 + \epsilon = -\frac{1}{2\kappa}\left(-2r' - \frac{3f^2}{2} + \frac{3r^2}{2} - ft\right),$$

$$\lambda = \frac{1}{\kappa}\left(\frac{f'}{2} + \frac{\dot{r}}{2} + \frac{fs}{4} - \frac{fr}{4} + \frac{tr}{4}\right).$$

Notice that these physical parameters depend only on 4 functions f, r, s, t, the precise ones which remain in the expression for $T_j{}^i{}_k$ if we examine unidirectionality in general $n+1$ dimensions where $n+1 \neq 4$.

The Jacobi identity (III.16.1) stands as is, namely $f' - \dot{r} + \frac{rf}{2} + \frac{tr}{2} - \frac{fs}{2} = 0$ and (2) can be written as $k' - \frac{ks}{2} - \dot{q} + q\,\frac{(f+t)}{2} + 2\dot{h} - h(f+t) - 2p' + ps = 0$ which can be satisfied, for instance, if $k = q = h = p \equiv 0$, though we need not impose this condition. It can be fulfilled in many ways without affecting the physical quantities σ_I, λ, $\rho c^2 + \epsilon$ above or the kinematic θ_{AB}, ω_{AB}, $\dot{u}_A$ which here depend only on f, r, s, t. Hence we can simply ignore (2) and the other Jacobi identities (4) through (7) are trivial in (III.16). Hence only the one equation is needed for the Jacobi identity, and we can use it to rewrite λ as $\lambda = \frac{1}{\kappa}\left(f' + \frac{tr}{2}\right)$. The integrability conditions are

$$\dot{f}' - f'^{\cdot} = \frac{1}{2}\left(\dot{f}s - f'(f+t)\right), \qquad \dot{r}' - r'^{\cdot} = \frac{1}{2}\left(\dot{r}s - r'(f+t)\right)$$

$$\dot{s}' - s'^{\cdot} = \frac{1}{2}\left(\dot{s}s - s'(f+t)\right), \qquad \dot{t}' - t'^{\cdot} = \frac{1}{2}\left(\dot{t}s - t'(f+t)\right).$$

As an important check on the accuracy of our calculations for the Ricci tensor and the Jacobi identity, we can substitute back into the equations

of conservation of energy and momentum namely (III.9.5) which are equivalent to the contracted Bianchi identity. We know this in general is implied by the Jacobi identity and integrability conditions. For our unidirectional Ricci coefficients with $p = g$ and $\hat{f} = h$ we have,

$$T_{4\,1}^{\;1} = \frac{f+t}{2}\,,\quad T_{4\,2}^{\;2} = T_{4\,3}^{\;3} = \frac{f}{2}\,,\quad T_{4\,I}^{\;I} = \frac{3f+t}{2}\quad \text{(sum on } I\text{)}$$

$$T_{1\,b}^{\;b} = T_1 = -\left(\frac{2r+s}{2}\right)\,,\quad T_{1\,4}^{\;4} = -\frac{s}{2}\,,\quad T_{1\,b}^{\;b} + T_{1\,4}^{\;4} = -(r+s)$$

$$T_{2\,4}^{\;4} = T_{3\,4}^{\;4} = 0,\quad T_{A\,b}^{\;b} = T_A = -\left(\frac{2r+s}{2}\right) r_A,\quad T_{A\,2}^{\;2} = T_{A\,3}^{\;3} = -\frac{r}{2} r_A\,,$$

$$T_{A\,1}^{\;1} = 0,\quad T_{A\,1}^{\;4} = 0,\quad T_{4\,A}^{\;1} = \frac{f+t}{2} r_A,\quad T_{4\,i}^{\;i} = T_4 = \frac{3f+t}{2}\,.$$

We can then put these values into (III.9.5) putting $\rho c^2 + \epsilon$ for ϵ because of mass conservation, and for λ we use the Jacobi simplified form $\lambda = \frac{1}{\kappa}\left(f' + \frac{tr}{2}\right)$. The result is an identity in each case, confirming the calculations as correct.

(III.19) The Dust Solutions.

The perfect fluid condition is $r' + s' - \dot{t} = -\frac{3ft}{2} + \frac{s^2}{2} - \frac{t^2}{2} - \frac{rs}{2}$ (i.e. $\sigma_1 = \sigma_2 = \sigma_3 = -P$) and the zero heat flow condition $\lambda = 0$ implies $f' + \frac{tr}{2} = 0$ and combining this with the Jacobi identity we have $\dot{r} = \frac{f(r-s)}{2}$. The condition $P = 0$ for dust is $2\dot{f} = \frac{3f^2}{2} - rs - \frac{r^2}{2}$. For a dust solution there is no work done by stress or heat flow, so $T^{ab} = -\rho c^2 u^a u^b$ and $T^{ab}{}_{;b} = 0$ implies $\dot{u}^a = 0$ so $s = 0$. Hence we have

$$f' = -\frac{tr}{2}\,,\quad \dot{r} = \frac{fr}{2}\,,\quad r' - \dot{t} = -\frac{3ft}{2} - \frac{t^2}{2}\,,\quad \dot{f} = \frac{3f^2}{4} - \frac{r^2}{4}\,.$$

Using these expressions we can check that $\dot{f}' - f'^{\cdot} = -\frac{1}{2}(f+t)f'$ holds identically, its value being $\frac{1}{4}(ftr + t^2 r)$. If we impose the condition $\dot{r}' - r'^{\cdot} = -\frac{1}{2}\, r'(f+t)$ we get a second order differential equation for

t involving time derivatives only, namely

$$\frac{5}{2} f\dot{t} + \frac{3}{2} t\dot{t} - \ddot{t} = \frac{5}{8} tr^2 + \frac{3}{8} tf^2 + \frac{5}{4} ft^2 + \frac{t^3}{4} \qquad \text{(III.19.1)}$$

The simplest way to solve this is to take $t = 0$ so we get $f' = r' = 0$ and our physical parameters have only time dependence even though a spatial direction $\underset{\sim}{r}$ is singled out. Therefore we have "concealed" spatial dependence and are very close to the original omnidirectional dust cosmologies. We obtain $\dot{r} = \frac{fr}{2}$, $\dot{f} = \frac{3f^2}{4} - \frac{r^2}{4}$ and $\rho c^2 + \epsilon = \frac{3}{4\kappa}(f^2 - r^2)$, so $r = 0$ is the Friedmann solution in (III.14) again in the special case $P = 0$, i.e. the parabolic solution of (III.13). Of course we may take $\epsilon = 0$ in $\rho c^2 + \epsilon$.

For $r \neq 0$ we can find a dust cosmology which is physically but not geometrically omnidirectional. It is easy to see that $\dot{\rho} = \frac{3}{2} f\rho$, so $\rho = c_0 r^3$ for a constant c_0. We may write $f^2 - r^2 = c_1 r^3$ for some constant c_1 so that $f = \mp r\sqrt{c_1 r + 1}$. Since the motion is irrotational and geodesic there is a local time coordinate τ which is proper time for each world line. Thus $\frac{dr}{d\tau} = -\dot{r} = -\frac{rf}{2} = \pm r^2\sqrt{c_1 r + 1}$ and integrating we find $\pm\frac{\tau}{2} = -\frac{\sqrt{c_1 r + 1}}{r} + c_1 \tanh^{-1}\left(\frac{1}{\sqrt{c_1 r + 1}}\right)$. If we take $c_1 > 0$, $r > 0$ and the minus sign above, we have an expanding universe that continues always to expand. This is the hyperbolic case mentioned in (III.13). Notice its similarity to the elliptical case.

(III.20) Only f and r Non-Zero.

As we have just done for the dust solution above, let us take f and r as the only functions which are non-zero in the unidirectional case, but allowing a pressure P this time. Again, $f' = 0$ and $r' = 0$ so the solution is physically omnidirectional. Furthermore

$\dot{r} = \frac{1}{2} rf$, $P = \frac{1}{4\kappa}(4\dot{f} - 3f^2 + r^2)$ and $\rho c^2 + \epsilon = \frac{3}{4\kappa}(f^2 - r^2)$. Compare these carefully with the corresponding results at the end of (III.12) if $\hat{f}$ is replaced by r and the signs are changed on $\hat{f}^2$. The duality holds even between $\dot{\hat{f}} = \frac{1}{2} f\hat{f}$ and $\dot{r} = \frac{1}{2} rf$. If $r = 0$ we get the same solution as taking $\hat{f} = 0$ in (III.12) which is also the one whose metric we obtained in (III.14).

(III.21) Unidirectional Flow-Static Solutions.

These solutions are of great interest in solving stable gravitational problems such as considered in (V.6). We impose the flow-static conditions $T^i_{j\ k,4} = 0$ and $T^I_{J\ 4} = 0$. Hence $\dot{f}$, $\dot{\hat{f}}$, $\dot{g}$, $\dot{h}$ etc. are all zero, and $0 = T^I_{J\ 4} \Rightarrow f = k = p = t = 0$. The integrability conditions (III.15.1) imply $h = \hat{f}$. All Jacobi identity equations are identically zero and are satisfied except (III.16.6) which becomes $2g' - 2gr + sg = 0$. The physical quantities are $\sigma_2 = \sigma_3 = -\frac{1}{2\kappa}\left(-r' - s' + \frac{r^2}{2} + \frac{s^2}{2} + \frac{g^2}{2} + \frac{rs}{2}\right)$, $\sigma_1 = -\frac{1}{2\kappa}\left(-\frac{g^2}{2} + \frac{r^2}{2} + rs\right)$, $\rho c^2 + \epsilon = -\frac{1}{2\kappa}\left(-2r' + \frac{3r^2}{2} - \frac{3g^2}{2}\right)$, and $\lambda = 0$. The solution is independent of the functions $h = \hat{f}$, and q.

The dust solution is easily obtained by putting $s = 0$ and $g = r$. We find $r' = r^2$ and $\rho c^2 = \frac{r'}{\kappa} = \frac{r^2}{\kappa}$. Other constitutive equations (than $P = 0$) can be imposed easily, for instance the radiative constitutive equation $P = \frac{1}{3}(\rho c^2 + \epsilon)$ and the high temperature relativistic ideal gas equation $P = \frac{1}{3}(\rho c^2 + \epsilon)$ (c.f. Chapter V).

There is a function x of position defined locally with $x_{,a} = r_a$ if and only if $dr_a|_b = 0$, i.e. $T_b{}^1{}_c = 0$. This holds automatically for our unidirectional flow-static case. Therefore, for the dust solution, absorbing the constant of integration into x, we have

$r = -\frac{1}{x}$, $\rho c^2 = \frac{1}{\kappa x^2}$.

(III.22) Weakly Flow-Static Unidirectional Cases.

For a unidirectional space-time we do not need the full force of the definition of flow-static to guarantee time independence to all orders of the derivatives of the functions in $T_{j\ k}^{\ i}$. It is enough to require the weakened conditions $T_{j\ k,4}^{\ i} = 0$ and $T_{1\ 4}^{\ 1} = 0$, since the functions already are independent of two spatial directions. In this case we have $f = t = 0$ while p and k may be non-zero, generalizing the solution in (III.21). Again, all time derivatives are zero, integrability implies $h = \hat{f}$, and the Jacobi identities in (III.16) give us the two equations $k' - 2p' - \frac{ks}{2} + ps = 0$ and $2g' - 2p' - 2gr + 2pr + sg - sp = 0$. From (III.17) we have $\sigma_2 = \sigma_3 = -\frac{1}{2\kappa}\left(-r' - s' + \frac{r^2}{2} + \frac{s^2}{2} + \frac{(g-p)^2}{2} + \frac{rs}{2}\right)$, $\sigma_1 = -\frac{1}{2\kappa}\left(\frac{1}{2}(p-g)(3p+g-2k) + \frac{r^2}{2} + rs\right)$, $\rho c^2 + \epsilon = -\frac{1}{2\kappa}\left(-2r' + \frac{3r^2}{2} + \frac{1}{2}(p-g)(3g+p-2k)\right)$, $\lambda = 0$.

Let us look for dust solutions. For these, the acceleration is zero, so $s = 0$ and $\sigma_1 = 0 \Rightarrow r^2 = (g-p)(3p+g-2k)$. Also $\sigma_2 = \sigma_3 = 0 \Rightarrow r' = \frac{r^2}{2} + \frac{(g-p)^2}{2}$, so differentiating the equation for r^2 we obtained by putting $\sigma_1 = 0$, using the Jacobi identities, and comparing with this latter equation we find the $\sigma_1 = 0$ condition automatically implies $\sigma_2 = \sigma_3 = 0$. Of course $g' - p' = (g-p)r$ and $k = 2p + c_0$ for a constant c_0. Hence $\rho c^2 + \epsilon = \frac{(g-p)^2}{\kappa}$ where

$g' - p' = (g-p)\sqrt{(g-p)(g-p-2c_0)}$. Thus $g - p$ is the only function the physics of this problem depends upon, and $c_0 = 0$ gives us the same type of solution as in (III.21) but with different Ricci coefficients. As in the previous section, we have a rigid geodesic motion with rotation. The motion is rigid ($\theta_{ab} = 0$) and hence the space-time is flow-stationary for any weakly flow-static unidirectional case, not

just the dust solution, since $f = t = 0$. Again $T_{b\ c}^{1} = 0$ so a local coordinate function x exists with $x_{,a} = r_a$ and putting $w = g - p$ we may write $w' = w\sqrt{w(w-2c_0)}$, $\rho c^2 = \frac{w^2}{\kappa}$. Of course $\frac{dw}{w\sqrt{w(w-2c_0)}} = dx$ can be integrated and we find $w = \frac{2c_0}{1 - c_0^2 x^2}$ where we have absorbed the constant of integration (which is the position shift) into the coordinate function x. This is for $c_0 \neq 0$, since $c_0 = 0$ was handled in (III.21). Here we find $\rho c^2 = \frac{4c_0^2}{\kappa(1 - c_0^2 x^2)^2}$ and we require $x > \frac{1}{c_0} > 0$ to avoid the singularity at $x = \frac{1}{c_0}$ and cover all physically interesting cases. The dust solutions here and in (III.21) are sustained by rotation i.e. vorticity, and in fact $\upsilon^1 = -\frac{w}{2}$ in this case. As in (III.21) the equations here are well adapted to the imposition of equations of state for non-zero pressure related to the mass and energy density.

(III.23) Directional Invariance - Generalization of Axisymmetry.

The symmetry conditions imposed on coordinate solutions in General Relativity have, as we have seen, counterparts in frame component solutions. The typical coordinate solutions conditions — static, stationary, homogeneous and isotropic, spherically symmetric — have the analogs of flow static, flow stationary, omnidirectional and unidirectional. The coordinate solutions are useful for an observer at infinity examining a system, which may have imposed boundary conditions such as being asymptotically flat. The frame component solutions are useful in continuum mechanics for an observer who rides along in the rest frame of the material medium examining what is happening locally. Here the physics of the material's constitutive properties are manifest, and we can work without reference to a coordinate system or even the frame component system — we need only the Ricci coefficients and their

derivatives to obtain all useful physical information. On the other hand boundary conditions and global properties are more difficult to establish. The concept in frame components which corresponds to axial symmetry in coordinates is directional invariance (along one spacelike direction).

A space-time is *directionally invariant* (along t^a) if the Ricci coefficients for an adapted frame component system satisfy $T_i{}^j{}_3 = 0$ for $j \neq 3$ and $T_j{}^i{}_{k,3} = 0$ $\forall_{i,j,k} = 1,2,3,4$. Similar results hold for other principal directions. Thus a directionally invariant set of Ricci coefficients satisfy $T_j{}^i{}_{k,\ell 3} = 0 = T_j{}^i{}_{k,\ell m3}$ etc. If we use directional invariance without reference to a principal unit vector, that vector is understood to be t^a. Clearly M is directionally invariant along u^a if and only if it is flow-static.

If more than one directional invariant symmetry is imposed we may weaken the restrictive condition on $T_j{}^i{}_k$ and still guarantee all orders of derivatives to be zero along these directions. For example if $T_j{}^i{}_k$ is to give weak directional invariance along s^a and t^a we have $T_j{}^i{}_{k,3} = 0 = T_j{}^i{}_{k,2}$ and $T_b{}^a{}_c = 0$ for $a,b = 1,4$ and $c = 2,3$. This holds in particular for a unidirectional space-time. Likewise if M is weakly directionally invariant along r^a, s^a and t^a then $T_j{}^i{}_{k,L} = 0$, and $T_4{}^4{}_J = 0$. A special case of this, is the omnidirectional space. Furthermore weak directional invariance along all 4 axes r^a, s^a, t^a, u^a is equivalent to $T_j{}^i{}_k$ being constant. This does not imply weak directional invariance along 3 axes.

(III.24) Unidirectionality and r^a - Directional Invariance.

If we combine the unidirectional condition with weak r^a directional invariance, then only time derivatives such as $\dot{f}$, $\dot{\hat{f}}$, $\dot{g}$, $\dot{h}$, $\dot{k}$ etc.

survive, and $T_{4\ J}^{\ 4} = 0$ implies $s = 0$ and the integrability conditions imply $p = g$ so the motion is irrotational and geodesic guaranteeing the existence of a local time coordinate τ on which all of our functions depend. The Jacobi identities from (III.16) are (1), $\dot{r} = \frac{r}{2}(f+t)$, (2), $-\dot{q} + 2\dot{h} - fh - 2\hat{f}t + \frac{qf}{2} + \frac{qt}{2} + th = 0$, (4), $2r(\hat{f}-h) = 0$, (5) and (6), trivial, (7), $2(\dot{h}-\dot{\hat{f}}) = (f-t)(h-\hat{f})$. We can solve these in one way by taking $\hat{f} = h$, $q = 2h$ and $\dot{r} = \frac{r}{2}(f+t)$. Then

$$\sigma_2 = \sigma_3 = -\frac{1}{2\kappa}\left(2\dot{f} + \dot{t} - \frac{3f^2}{2} + \frac{r^2}{2} - \frac{3ft}{2} - \frac{t^2}{2}\right),$$

$$\sigma_1 = -\frac{1}{2\kappa}\left(2\dot{f} + \frac{r^2}{2} - \frac{3f^2}{2}\right), \quad \rho c^2 + \epsilon = -\frac{1}{2\kappa}\left(-\frac{3f^2}{2} + \frac{3r^2}{2} - ft\right),$$

$$\lambda = \frac{tr}{2\kappa}.$$

Then if $t = 0$ we have the physically omnidirectional solution of (III.20), so $t \neq 0$ generalizes this to non-zero shear. The equal stress condition $\dot{t} = \frac{3ft + t^2}{2}$ puts a constraint equation on the shear t for a perfect fluid, which is, of course, satisfied for $t \equiv 0$.

Alternatively, we may put $r = 0$ and $\hat{f} \neq h$ so that we have the differential equations $\dot{q} = 2\dot{h} - fh - 2\hat{f}t + \frac{qf}{2} + \frac{qt}{2} + th$ and $2(\dot{h}-\dot{\hat{f}}) = (f-t)(h-\hat{f})$. Then

$$\sigma_2 = \sigma_3 = -\frac{1}{2\kappa}\left(2\dot{f} - \frac{3f^2}{2} + \dot{t} - \frac{3ft}{2} + \frac{t^2}{2} - \frac{(h-\hat{f})^2}{2}\right),$$

$$\sigma_1 = -\frac{1}{2\kappa}\left(2\dot{f} - \frac{3f^2}{2} + \frac{1}{2}(h-\hat{f})(3h+\hat{f}-2q)\right), \quad \lambda = 0,$$

$$\rho c^2 + \epsilon = -\frac{1}{2\kappa}\left(-\frac{3f^2}{2} + \frac{1}{2}(h-\hat{f})(h+3\hat{f}-2q) - ft\right).$$

This reduces to the omnidirectional solution (III.12) for $q = h = t = 0$, and therefore generalizes it to include, among other things, the shear t.

If we have zero shear ($t = 0$) we expect, even for a viscous fluid, that the perfect fluid condition $\sigma_1 = \sigma_2 = \sigma_3 = -P$ should hold.

If $h - \hat{f} \neq 0$ we see this implies $q = 2h$ which also holds in the Jacobi identity for q if $t = 0$. Putting $w = h - \hat{f}$ we see that $\dot{w} = \frac{1}{2} fw$ and $\sigma_2 = \sigma_3 = -\frac{1}{2\kappa}\left(2\dot{f} - \frac{3f^2}{2} - \frac{w^2}{2}\right)$, $\sigma_1 = -\frac{1}{2\kappa}\left(2\dot{f} - \frac{3f^2}{2} - \frac{w^2}{2}\right)$, $\lambda = 0$, $\rho c^2 + \epsilon = -\frac{1}{2\kappa}\left(-\frac{3f^2}{2} - \frac{3w^2}{2}\right)$. Comparing this to the omnidirectional solutions in (III.12) we see they are identical if w is replaced by $\hat{f}$. Therefore we have here too, a generalization of the omnidirectional solution for the case $t \neq 0$.

(III.25) Particular Motions in Frame Components.

In order to examine the notion of a motion of constant stretch history for a unidirectional space-time (I.30) we need to express the convective derivative (I.8) in frame components as is done for the Lie derivative in (II.20). From these results,

$$L_{\underset{\sim}{u}} E^k{}_m = E^k{}_{m,\ell} u^\ell - E^\ell{}_m u^k{}_{,\ell} + E^k{}_\ell u^\ell{}_{,m} + E^j{}_m T_j{}^k{}_\ell u^\ell - E^k{}_j T_m{}^j{}_\ell u^\ell ,$$

and

$$\mathcal{D}E^k{}_m = L_{\underset{\sim}{u}} E^k{}_m - E^\ell{}_m u^k u_\ell + E^k{}_\ell u^\ell u_m .$$

For $E^k{}_\ell$ orthogonal we can drop the $E^k{}_\ell u^{\ell\cdot} u_m$ term from here, and then put $E^k{}_\ell = W^k{}_\ell + \theta^k{}_\ell$ where $W_{k\ell} = \varepsilon_{k\ell m} v^m$. For the unidirectional case, we require $W_{k\ell} = w\epsilon_{k\ell m} r^m = w\Lambda_{k\ell}$ in frame components for some function w with $\dot{w}$ and w' derivatives only. Substituting and setting $\mathcal{D}E^k{}_m = 0$ we find that $0 = E^k{}_{m,4} + E^j{}_m T_j{}^k{}_4 - E^k{}_j T_m{}^j{}_4 - E^\ell{}_m u^{k\cdot} u_\ell$ where

$$E^k{}_m = \frac{f}{2}\gamma^k_m + \frac{t}{2} r^k r_m + w\Lambda^k{}_m, \quad E^k{}_{m,4} = -\frac{\dot{f}}{2}\gamma^k_m - \frac{\dot{t}}{2} r^k r_m - \dot{w}\Lambda^k{}_m .$$

Then we can easily see that $E^\ell{}_m u^{k\cdot} u_\ell = u^k r_m\left(-\frac{fs}{4} - \frac{st}{4}\right)$ and $E^k{}_j T_m{}^j{}_4 = \gamma^k_m\left(\frac{wk}{2} - wp - \frac{f^2}{4}\right) + \Lambda^k{}_m\left(\frac{fp}{2} - \frac{fk}{4} - \frac{wf}{2}\right) + r_k r_m\left(wp - \frac{ft}{2} - \frac{t^2}{4} - \frac{wk}{2}\right)$ and $E^j{}_m T_j{}^k{}_4 = E^k{}_j T_m{}^j{}_4 + E^\ell{}_m u^{k\cdot} u_\ell$. Therefore the condition for a motion

of constant stretch history in the unidirectional case becomes $\dot{f} = 0$, $\dot{t} = 0$, i.e. f and t are constants along flow lines. The condition for a motion to be a viscometric flow $E^k{}_m E^m{}_\ell = 0$ (I.30) is that it be rigid in the unidirectional case, i.e. $f = t = 0$. We see that all unidirectional flow-static solutions describe motions of constant stretch history, in fact even weakly flow-static solutions (III.22) are viscometric flows. Of course also $\dot{w} = 0$ so w is a constant along flow lines.

A motion is isochoric if $3f + t = 0$ and of zero acceleration or geodesic if $s = 0$. We say a motion is of *constant acceleration* if the acceleration vector is Christoffel symbol Fermi transported along the flow lines, i.e. $\dot{u}^a{}_{;b} u^b = u^a(\dot{u}_b \dot{u}^b)$. For the unidirectional case it is easy to show that this holds if and only if $\dot{s} = 0$, so s is a constant along flow lines. The *shear tensor* defined by $\sigma_{ab} = \theta_{ab} - \frac{\theta}{3}\gamma_{ab}$ has zero trace $\sigma^a{}_a = 0$ and is given, for a unidirectional space-time by $\sigma_{ab} = \frac{t}{2} r_a r_b - \frac{t}{6}\gamma_{ab}$ so $\sigma_{ab}\sigma^{ab} = \frac{t^2}{6}$. If $t = 0$ the motion is shear-free and the expansion is isotropic.

(III.26) Proposed Bidirectional Solutions.

In order to solve the problem of rotating stars and galaxies and other such phenomena, we need to examine a set of Ricci coefficients to cover a weakly flow static directionally invariant case. If two spatial directions are singled out we say the Ricci coefficients are *bidirectional*. In the case $n + 1 = 4$ this is the most general case. If $n + 1$ is completely general, then $T_b{}^a{}_c$ is a linear combination of $\delta^a{}_{[b}u_{c]}$, $\delta^a{}_{[b}r_{c]}$, $\delta^a{}_{[b}s_{c]}$, $u^a u_{[b}r_{c]}$, $u^a r_{[b}s_{c]}$, $u^a u_{[b}s_{c]}$, $r^a u_{[b}r_{c]}$, $r^a r_{[b}s_{c]}$, $r^a u_{[b}s_{c]}$, $s^a u_{[b}r_{c]}$, $s^a r_{[b}s_{c]}$, $s^a u_{[b}s_{c]}$. The Ricci tensor

R_{jk} will be in the form of adapted frame components except for a single term of the form $(s_j t_k + t_j s_k)$ which can be set to zero giving one constraint equation. Of course all functions which are coefficients must have u^a, r^a and s^a directed derivatives only and none along the other $n - 2$ space-like principal directions.

Putting $n + 1 = 4$ in the unidirectional case, if we ignore the volume element determined terms we have only four functions f, r, s, t instead of the regular ten. Likewise in the bidirectional case we have only 12 functions (one for each of the aforementioned terms) instead of the complete 24. If we impose the condition that the Ricci coefficients are weakly flow-static and directionally invariant, then $T_j{}^i{}_k = 0$, $i,j = 1,2$, $k = 3,4$, and so we have that (for $n + 1 = 4$),

$$T_b{}^a{}_c = ft^a t_{[b}u_{c]} + gt^a t_{[b}r_{c]} + ht^a t_{[b}s_{c]} + \mathcal{k}u^a u_{[b}r_{c]}$$
$$+ \mathcal{p}u^a r_{[b}s_{c]} + \mathcal{q}u^a u_{[b}s_{c]} + \mathcal{r}r^a r_{[b}s_{c]} + \mathcal{s}s^a r_{[b}s_{c]}$$

where all functions have only 1 or 2 derivatives, i.e. $f_{,3} = f_{,4} = g_{,3} = g_{,4} = \cdots = 0$. We have replaced $\delta^a_{[b}u_{c]}$ by $t^a t_{[b}u_{c]}$ and $\delta^a_{[b}r_{c]}$ by $t^a t_{[b}r_{c]}$ etc. because of the combination degeneracy for $n + 1 = 4$ with other terms, then eliminated the unneeded ones using the directional invariance condition. The higher dimensional generalization case for the unidirectional solutions contained most of the ones of interest and the simplest Jacobi identity and integrability conditions, and we found it is (III.18) simply by putting $\hat{f} = h$ and $g = \mathcal{p}$ which left us with only the four functions f, r, s, t of physical significance. Likewise we could expect to capture the most important solutions in the bidirectional case this way, limiting the amount of

calculation. We also have the advantage here of having only one equation instead of three to guarantee an adapted frame component system, i.e. R_{jk} diagonal in the orthogonal part. Working out the kinematical parameters we find that $\nu^3 = -\omega_{12} = -\frac{p}{4}$, $\nu^1 = \nu^2 = 0$ is the vorticity, and $\dot{u}_A = \frac{k}{2} r_A + \frac{q}{2} s_A$ and $\theta_{AB} = \frac{f}{2} t_A t_B$, from (III.9.4). For a flow stationary solution we must have rigid motion and so $f = 0$. These Ricci coefficients are being designed to solve the problem of spherical stars, axisymmetric rotating stars and rotating galaxies. In the usual spherical polar coordinates (r,θ,ϕ), the r direction is radial and corresponds closely to r^a as θ direction does to s^a and ϕ direction to t^a. We shall see in (V.6) that the Schwarzschild solution can be placed in frame components of this form and thereby provides a way to match exterior vacuum solutions for spherical stars to interior solutions.

Notice however that the vorticity appears only along the $\underset{\sim}{t}$ direction, corresponding to ϕ in polar coordinates which is not what we would expect for rotating stars. Thus we see even now, the bi-directional condition is too strong for the rotating case, where we have to examine the general weakly flow-static directionally invariant case which leaves us with 16 functions of two variables to determine. This can be reduced somewhat by putting $\theta_{AB} = 0$, but the problem of solving the rotating stars with say an exterior Kerr solution converted to frame components is left for (V.10).

Returning to the bidirectional case, we see that if $f = 0$ then we have the Ricci coefficients (strongly) flow-static and directionally invariant. If we demand the existence of a radial coordinate we obtain $r = 0$, a basic condition for spherical symmetry. If the gravitational acceleration is radially directed then $q = 0$ and similarly zero

vorticity means $p = 0$ which we must have if our solution is physically unidirectional. Thus, combining all these results, we have the Ricci coefficients

$$T_b{}^a{}_c = gt^a t_{[b} r_{c]} + ht^a t_{[b} s_{c]} + k u^a u_{[b} r_{c]} + s s^a r_{[b} s_{c]} \qquad \text{(III.26.1)}$$

for the spherically symmetric case. Here g, h, k, s have only r^a and s^a directed derivatives, and we write $g_{,\ell} = g_1 r_\ell + g_2 s_\ell$, $h_{,\ell} = h_1 r_\ell + h_2 s_\ell$ and similarly for k and s. We write g_{12} for $g_{,12}$ and similarly for other derivatives. We then find the integrability conditions are

$$\begin{aligned} g_{12} - g_{21} &= \frac{s}{2} g_2, \quad h_{12} - h_{21} = \frac{s}{2} h_2 , \\ k_{12} - k_{21} &= \frac{s}{2} k_2, \quad s_{12} - s_{21} = \frac{s}{2} s_2 . \end{aligned} \qquad \text{(III.26.2)}$$

The Jacobi identity implies

$$g_2 - h_1 = \frac{hs}{2} \quad \text{and} \quad k_2 = 0, \qquad \text{(III.26.3)}$$

and the Ricci tensor calculation gives us the conditions

$$g_2 = \frac{h}{2}(g+s) \quad \text{and} \quad h_1 = \frac{gh}{2} \qquad \text{(III.26.4)}$$

for adapted frame components $(R_{12} = 0)$ as well as

$$\begin{aligned} \sigma_1 &= -\frac{1}{2\kappa}\left(-h_2 + \frac{h^2}{2} - \frac{gk}{2} - \frac{gs}{2} + \frac{ks}{2}\right), \\ \sigma_2 &= -\frac{1}{2\kappa}\left(k_1 - g_1 + \frac{g^2}{2} + \frac{k^2}{2} - \frac{gk}{2}\right), \\ \sigma_3 &= -\frac{1}{2\kappa}\left(k_1 + s_1 + \frac{s^2}{2} + \frac{k^2}{2} + \frac{sk}{2}\right), \quad \lambda = 0, \\ \rho c^2 + \epsilon &= -\frac{1}{2\kappa}\left(s_1 - g_1 - h_2 + \frac{g^2}{2} + \frac{s^2}{2} + \frac{h^2}{2} - \frac{sg}{2}\right). \end{aligned} \qquad \text{(III.26.5)}$$

Now $\sigma_2 = \sigma_3$ implies

$$s_1 + g_1 = \frac{g^2}{2} - \frac{gk}{2} - \frac{s^2}{2} - \frac{sk}{2} \, , \qquad \text{(III.26.6)}$$

and $\sigma_1 = \sigma_2$ implies (for isotropic pressure)

$$k_1 + h_2 - g_1 = \frac{h^2}{2} - \frac{g^2}{2} - \frac{k^2}{2} - \frac{gs}{2} + \frac{ks}{2} \; . \qquad \text{(III.26.7)}$$

Testing the condition $\sigma_{2,2} = 0$ of physical unidirectionality we find it automatically holds using (2), (3), (4) and (6) above. In (V.7) we shall use these results to examine spherical stars. If we impose the physical unidirectional condition $(\rho c^2 + \epsilon)_{,2} = 0$ then we see that using (2), (3), (4), (6) and (7) it is equivalent to $ks_2 = \frac{kh}{2}(g+s)$, or for $k \neq 0$ we have

$$s_2 = g_2 = \frac{h}{2}(g+s). \qquad \text{(III.26.8)}$$

We now differentiate (6) along the 2-direction and substitute from (2), (3), (4), (6) and (8) to obtain $0 = h(g+s)^2$. Hence for $h \neq 0$ we have

$$g + s = 0, \quad s_2 = g_2 = 0, \qquad \text{(III.26.9)}$$

and (6) becomes a triviality. The choices of $k \neq 0$ and $h \neq 0$ for (8) and (9) will be seen as necessary when we consider the Schwarzschild solution and boundary matching to the interior solution in (V.6) and (V.7).

Finally if we take the 2-derivative of the expression for h_1 in (4) and apply (2) and (9) we obtain

$$h_{21} = g\, h_2 \qquad \text{(III.26.10)}$$

which is the basic differential equation for h_2. Also taking the 2-derivative of (7) we have

$$h_{22} = h\, h_2 \; . \qquad \text{(III.26.11)}$$

This completes the basic relations we need for the case of spherical symmetry in the flow-static directionally invariant bi-directional case. The advantages of the method of Ricci coefficients, using the Jacobi identity, integrability conditions and conditions on R_{jk} to find specific solutions under specified energy-momentum tensor restrictions are obvious. We simply have to differentiate, substitute and solve repeatedly to reduce the problem.

CHAPTER IV

A GENERAL THEORY FOR A SMOOTH MATERIALLY UNIFORM THERMODYNAMIC SIMPLE BODY

(IV.1) The Thermodynamic Material Element.

Following the notation and approach of Noll [77] we let (c.f. (P.9)) $\mathcal{T}$ be an n-dimensional real vector space, and $\mathrm{Sym}(\mathcal{T},\mathcal{T}^*)$ be the vector space of symmetric real bilinear forms on $\mathcal{T}$, and $\mathrm{Sym}^+(\mathcal{T},\mathcal{T}^*)$ the positive definite ones. We let $\mathrm{Lin}(A,B)$ denote the set (a vector space) of linear maps from the vector space A to the vector space B. Configuration space $\mathcal{G}$ is defined to be a closed and connected submanifold of $\mathrm{Sym}^+(\mathcal{T},\mathcal{T}^*) \times \mathrm{Lin}\big(\mathcal{T}, \mathrm{Sym}(\mathcal{T},\mathcal{T}^*)\big) \times \mathbb{R}^+ \times \mathcal{T}^* \times \mathcal{T}$. An element $G \in \mathcal{G}$ is written as $G = (D,H,\Theta,g,\dot{U})$, where D is interpreted as material deformation, H as spatial gradient of deformation, Θ as temperature, g as thermal gradient, and $\dot{U}$ as the acceleration of the material medium. Because of the vector space structure, we can canonically identify $\mathcal{G}_G$ the tangent space to $\mathcal{G}$ at G with a vector subspace of $\mathrm{Sym}(\mathcal{T},\mathcal{T}^*) \times \mathrm{Lin}\big(\mathcal{T}, \mathrm{Sym}(\mathcal{T},\mathcal{T}^*)\big) \times \mathbb{R} \times \mathcal{T}^* \times \mathcal{T}$. If $v_G \in \mathcal{G}_G$ we write $v_G = (\partial D, \partial H, \partial\Theta, \partial g, \partial\dot{U})$ as its elements.

We shall assume the existence of a differential manifold Σ called the *state manifold* and a *configuration map* $\hat{G}\colon \Sigma \to \mathcal{G}$ which is differentiable. From this we can easily obtain a vector bundle structure for $\left\{\left(\sigma, v_{\hat{G}(\sigma)}\right) \middle| \sigma \in \Sigma,\ v_{\hat{G}(\sigma)} \in \mathcal{G}_{\hat{G}(\sigma)}\right\}$ over the base space Σ. Suppose that for each $G \in \mathcal{G}$ a submanifold $\tilde{\mathcal{G}}_G$ of the vector space $\mathcal{G}_G$ is selected such that the set $E\Sigma = \left\{\left(\sigma, v_{\hat{G}(\sigma)}\right) \middle| \sigma \in \Sigma,\right.$ $\left. v_{\hat{G}(\sigma)} \in \tilde{\mathcal{G}}_{\hat{G}(\sigma)}\right\}$ together with the natural projection π into the base

space Σ is a fibration. (See Dieudonné [22], p. 77). Then we call $\tilde{G}_G$ the *manifold of consistent vectors* at G and $E\Sigma$ the *evolution fibre bundle* of the state space Σ.

We assume there exists a smooth map $\hat{\rho}: E\Sigma \to T\Sigma$ called the *evolution function*, where $T\Sigma$ is the tangent bundle of the state space. It satisfies $\pi = \pi' \circ \hat{\rho}$ and $\lambda = \hat{G}_* \circ \hat{\rho}$ where

$$\begin{tikzcd} E\Sigma \arrow[r, "\hat{\rho}"] \arrow[d, "\pi"] & T\Sigma \arrow[d, "\pi'"] \\ \Sigma \arrow[r, "1_\Sigma"'] & \Sigma \end{tikzcd} \qquad \begin{tikzcd} E\Sigma \arrow[r, "\hat{\rho}"] \arrow[dr, "\lambda"'] & T\Sigma \arrow[d, "\hat{G}_*"] \\ & TG \end{tikzcd}$$

$\lambda\left(\sigma, v_{\hat{G}(\sigma)}\right) = v_{\hat{G}(\sigma)} \in \tilde{G}_{\hat{G}(\sigma)} \subset G_{\hat{G}(\sigma)}$ and $\hat{G}_*$ is the derivative of $\hat{G}$.

Let $\sigma \in \Sigma$ be a state with $G = \hat{G}(\sigma) \in G$ its configuration. Let $P: [0, d_p] \to G$ be a smooth curve with $P(0) = G$ and $d_p > 0$, such that $\left.\frac{dP}{dt}\right|_t = \dot{P}(t) \in \tilde{G}_{P(t)}$ for all $t \in [0,d_p]$. We call such a map P a *smooth process* in G. We say P_0 is a (general) *process* if P_0 is a finite continuation $P_0 = P_1 * P_2 * \cdots * P_m$ of smooth processes as defined (P.9) by Noll [77] p. 10. Thus P_0 is continuous and piecewise smooth, and even at points of discontinuity of $\dot{P}_0$, the left and right hand derivatives with respect to t are consistent vectors.

A basic lifting theorem of differential geometry tells us that corresponding to our smooth process P there is a unique lifted process in the state manifold, which is a smooth curve $Q: [0,d_p] \to \Sigma$ with $Q(0) = \sigma$, $\hat{G} \circ Q(t) = P(t)$, $\hat{\rho}\left(Q(t), \dot{P}(t)\right) = \dot{Q}(t)$ $\forall\, t \in [0,d_p]$. Likewise, if $P_1(0) = \sigma_1$ the process P_0 has a lifted process Q_0 in Σ with $Q_0(0) = \sigma_1$, Q_0 being continuous and piecewise smooth.

As Noll has done [77, p. 10] we can define special processes called freezes (if the zero vector is consistent at each configuration), and segments of processes, and consider a class Π of processes satisfying his axioms P1 through P4. Later on we shall see how the first and second laws of thermodynamics restrict the class of admissible processes [77, p. 48].

We define the stress vector space $\mathcal{S} = \mathrm{Sym}(\mathcal{T}^*, \mathcal{T}) \times \mathbb{R}^2$, and an element $R \in \mathcal{S}$ is written as $R = (S, \eta, \psi)$ where S is intrinsic stress, η is entropy per unit volume and ψ is (Helmholtz) free energy density. We assume the existence of a smooth map $\hat{S}: \Sigma \to \mathcal{S}$ called the (generalized) *stress function*, and a map $\hat{H}: E\Sigma \to \mathcal{T}$ which is also differentiable and is called the *heat flux function*. By defining $\hat{H}$ on the evolution manifold rather than the state space, we are able to avoid the problem of infinite velocity of heat conduction associated with Fourier's law discussed by Maugin [58], Müller [74] and others.

In order to limit the size of the state space Σ we introduce an axiom for determining the state by stress which corresponds to Noll's Axiom III.

State determination Axiom: *If* $\sigma, \sigma' \in \Sigma$ *are such that* $\hat{G}(\sigma) = \hat{G}(\sigma') = G \in \mathcal{G}$ *and* $\hat{S}(\sigma) = \hat{S}(\sigma')$ *and furthermore* $\hat{S}_* \circ \hat{\rho}(\sigma, v_G) = \hat{S}_* \circ \hat{\rho}(\sigma', v_G)$ *for all* $v_G \in \tilde{\mathcal{G}}_G$ *then* $\sigma = \sigma'$.

Notice that since $\mathcal{S}$ is a vector space $T\mathcal{S} = \mathcal{S} \times \mathcal{S}$ canonically and $\hat{S}_*: T\Sigma \to \mathcal{S} \times \mathcal{S}$. This axiom also tells us that a finite dimensional state space will suffice.

The accessibility axiom of Noll can also be carried over here in slightly modified form.

Accessibility Axiom. *There exists a state* $\sigma_0 \in \Sigma$ *with configuration* $G_0 = \hat{G}(\sigma_0)$ *such that for any* $\sigma \in \Sigma$ *there exists a configuration process* $P\colon [0,d_p] \to G$ *with* $P(0) = G_0$ *whose tangent vectors* $\dot{P}(t)$ *are consistent for each* t *and for which the lifted curve* $Q\colon [0,d_p] \to \Sigma$ *with* $Q(0) = \sigma_0$ *satisfies* $Q(d_p) = \sigma$.

This axiom implies, among other things, that the state manifold Σ is connected. In order that all configurations correspond to a state, we assume $\hat{G}$ is surjective. Clearly the image $\hat{S}(\Sigma)$ is a connected set in S, and we assume, furthermore that it is a submanifold of S.

A *Thermodynamic Material Element* is a 10-tuple $\mathcal{E} = (T,G,\Sigma,S,E\Sigma,\hat{G},\hat{S},\hat{H},\hat{\rho},m)$ subject to the conditions above, with only m to be defined yet, plus some further restrictions.

(IV.2) The Mass Element.

For T an n-dimensional vector space and T^* its dual, the symbols $\overset{n}{\Lambda}T$ and $\overset{n}{\Lambda}T^*$ represent n-fold exterior products and are each one dimensional vector spaces. If $v = (\underset{\sim}{v}_{(1)},\cdots,\underset{\sim}{v}_{(n)})$ is a basis for T and $v^* = (\underset{\sim}{v}^{(1)*},\cdots,\underset{\sim}{v}^{(n)*})$ is the dual basis for T^* then

$\varepsilon_v = \underset{\sim}{v}_{(1)} \wedge \cdots \wedge \underset{\sim}{v}_{(n)}$ is a non-zero element of $\overset{n}{\Lambda}T$, and

$\varepsilon_v^* = \underset{\sim}{v}^{(1)*} \wedge \cdots \wedge \underset{\sim}{v}^{(n)*}$ is a non-zero element of $\overset{n}{\Lambda}T^*$. We have

$\langle \underset{\sim}{v}^{(i)*}, \underset{\sim}{v}_{(j)} \rangle = \delta^i_j$ and $\langle \varepsilon_v^*, \varepsilon_v \rangle = \dfrac{1}{n!}$.

If an element $D \in \mathrm{Sym}^+(T,T^*)$ is specified, an inner product and hence a norm is given for T. A basis $v = (\underset{\sim}{v}_{(1)},\cdots,\underset{\sim}{v}_{(n)})$ is said to be *orthonormal* with respect to D if $\langle \underset{\sim}{v}_{(j)}, D(\underset{\sim}{v}_{(i)}) \rangle = \delta^i_j$, i.e. $D(\underset{\sim}{v}_{(i)}) = v^{(i)*}$ $i = 1,\cdots,n$.

If $v = (\underset{\sim}{v}_{(i)})$, $v' = (\underset{\sim}{v}'_{(i)})$, $1 \le i \le n$ are orthonormal bases

for $\mathcal{T}$ corresponding to D then $\varepsilon_v = \pm\varepsilon_{v'}$, $\varepsilon^*_v = \pm\varepsilon^*_{v'}$. Thus, up to sign, a metric determines a volume element. Also ε^*_v is obtained naturally from ε_v by "lowering indices" using "the metric D", that is, the natural transformation from contravariant to covariant tensors using D.

On a material element it is assumed that a distinguished non-zero n-covector $m \in \overset{n}{\Lambda}\mathcal{T}^*$ is selected which is called the *rest mass* of the material element. If a deformation $D \in \mathrm{Sym}^+(\mathcal{T},\mathcal{T}^*)$ is chosen for our thermodynamic element, and we pick an orthonormal basis v for $\mathcal{T}$ corresponding to D then we can write $m = \rho\varepsilon^*_v = \rho V$. If $\rho > 0$ we say that the basis v is *oriented* and call the scalar ρ the *density* and $V = \varepsilon^*_v$ the *volume* occupied by the material element. The density is independent of the choice of basis.

(IV.3) Material Isomorphisms of Thermodynamic Material Elements.

Let $\mathcal{E} = (\mathcal{T},\mathcal{G},\Sigma,\mathcal{S},E\Sigma,\hat{G},\hat{S},\hat{H},\hat{\rho},m)$ and $\mathcal{E}' = (\mathcal{T}',\mathcal{G}',\Sigma',\mathcal{S}',E\Sigma',\hat{G}',\hat{S}',\hat{H}',\hat{\rho}',m')$ be two thermodynamic material elements. If $n = n'$ and $\Phi\colon \mathcal{T} \to \mathcal{T}'$ is a linear isomorphism, we will proceed to define what is meant by saying Φ is a material isomorphism, $(n' = \dim \mathcal{T}')$.

First Φ must induce a diffeomorphism $\Phi_{\mathcal{G}}\colon \mathcal{G} \to \mathcal{G}'$ by taking $G = (D,H,\Theta,g,\dot{U})$ to $G' = (D',H',\Theta',g',\dot{U}')$ where $D' = \Phi^{*-1} \circ D \circ \Phi^{-1}$, $H' = \Phi^{*-1} \circ H \circ (\Phi^{-1} \times \Phi^{-1})$, $\Theta' = \Theta$, $g' = \Phi^{*-1}(g)$, $\dot{U}' = \Phi(\dot{U})$, and where $\Phi^*\colon \mathcal{T}'^* \to \mathcal{T}^*$ is the dual isomorphism, and the map $H\colon \mathcal{T} \to \mathrm{Sym}(\mathcal{T},\mathcal{T}^*)$ is written as $H\colon \mathcal{T} \times \mathcal{T} \to \mathcal{T}^*$, the first component of the product being the original domain.

Next Φ must induce a diffeomorphism between the images $\hat{S}(\Sigma)$ and $\hat{S}'(\Sigma')$ in the stress spaces (which were assumed to be submanifolds of $\mathcal{S}$ and $\mathcal{S}'$ respectively). We call this diffeomorphism $\Phi_{\mathcal{S}}\colon \hat{S}(\Sigma) \to \hat{S}'(\Sigma')$.

It is defined by $\Phi_S(R) = R'$ where $R = (S,\eta,\psi)$, $R' = (S',\eta',\psi')$ and $S' = \Phi \circ S \circ \Phi^*$, $\eta' = \eta$ and $\psi' = \psi$.

Furthermore, we require that there exists a diffeomorphism $\Phi_\Sigma\colon \Sigma \to \Sigma'$ such that the two diagrams shown below commute.

$$\begin{tikzcd} \Sigma \arrow[r, "\Phi_\Sigma"] \arrow[d, "\hat{G}"] & \Sigma' \arrow[d, "\hat{G}'"] \\ G \arrow[r, "\Phi_G"] & G' \end{tikzcd} \qquad \begin{tikzcd} \Sigma \arrow[r, "\Phi_\Sigma"] \arrow[d, "\hat{S}"] & \Sigma' \arrow[d, "\hat{S}'"] \\ \hat{S}(\Sigma) \arrow[r, "\Phi_S"] & \hat{S}'(\Sigma') \\ \cap & \cap \\ S & S' \end{tikzcd}$$

We require also that $m' = \left(\overset{n}{\Lambda}\Phi^{*-1}\right)(m)$ where $\overset{n}{\Lambda}\Phi^{*-1}$ is the naturally induced linear map on n-covectors induced from Φ. Now we have a naturally induced diffeomorphism $\Phi_{E\Sigma}\colon E\Sigma \to E\Sigma'$ on the evolution bundles defined by $\Phi_{E\Sigma}\left(\sigma, v_{\hat{G}(\sigma)}\right) = \left(\Phi_\Sigma(\sigma), \Phi_{G*}(v_{\hat{G}(\sigma)})\right)$. We require that the evolution map diagram should commute,

$$\begin{tikzcd} E\Sigma \arrow[r, "\Phi_{E\Sigma}"] \arrow[d, "\hat{\rho}"] & E\Sigma' \arrow[d, "\hat{\rho}'"] \\ T\Sigma \arrow[r, "\Phi_{E*}"] & T\Sigma' \end{tikzcd}$$

and finally the heat flux functions should satisfy $\hat{H}' = \Phi \circ \hat{H} \circ \Phi_{E\Sigma}^{-1}$. When all these conditions are satisfied, then Φ is a material isomorphism from $\mathcal{E}$ to $\mathcal{E}'$.

If $\Phi\colon \mathcal{T} \to \mathcal{T}'$ and $\Psi\colon \mathcal{T}' \to \mathcal{T}''$ are material isomorphisms then so are $\Psi \circ \Phi$, Φ^{-1}, Ψ^{-1}. Furthermore, for any material isomorphism $\Phi\colon \mathcal{T} \to \mathcal{T}'$ the state space diffeomorphism Φ_Σ is unique. If not, then by composing Φ with Φ^{-1} we have the identity $I\colon \mathcal{T} \to \mathcal{T}$ a material isomorphism whereas the corresponding map $I_\Sigma\colon \Sigma \to \Sigma$ can be chosen

different from the identity 1_Σ. If $I_\Sigma(\sigma) = \sigma'$, $\sigma \neq \sigma'$ then $\hat{G}(\sigma) = \hat{G}(\sigma')$ and $\hat{S}(\sigma) = \hat{S}(\sigma')$ since Φ_G and Φ_S are identity maps. Furthermore from our commutative diagrams $\hat{S}_* \circ \rho(\sigma, v_G) = \hat{S}_* \circ \rho(\sigma', v_G)$ where $G = \hat{G}(\sigma) = \hat{G}(\sigma')$ and $v_G \in \tilde{G}_G$. Therefore, by our axiom for state determinism, $\sigma = \sigma'$ and in fact we require that $I_\Sigma = 1_\Sigma$ is the identity diffeomorphism. Therefore induced diffeomorphisms on the state spaces are unique for any material isomorphism between thermodynamic material elements.

$$\begin{tikzcd}
E\Sigma \arrow[r, "I_{E\Sigma}"] \arrow[d, "\hat{\rho}"] & E\Sigma \arrow[d, "\hat{\rho}"] \\
T\Sigma \arrow[r, "I_{\Sigma *}"] \arrow[d, "\hat{S}_*"] & T\Sigma \arrow[d, "\hat{S}_*"] \\
T\hat{S}(\Sigma) \arrow[r, "\Phi_{S*}", "(=\text{ identity})"'] & T\hat{S}(\Sigma)
\end{tikzcd}$$

The set of all material isomorphisms $\Phi\colon \mathcal{T} \to \mathcal{T}$ from a material element $\hat{\mathcal{E}}$ to itself forms a group called the *symmetry group* $g(\hat{\mathcal{E}})$ of the material element $\hat{\mathcal{E}}$. The condition $m = (\overset{n}{\Lambda}\Phi^{*-1})(m)$ for $\Phi \in g(\hat{\mathcal{E}})$ implies $g(\hat{\mathcal{E}})$ is a subgroup of $SL(\mathcal{T})$, the special linear group of those linear transformations of determinant $+1$.

We are now ready for the final basic axioms on our material element to complete its definition. We assume first that $g(\hat{\mathcal{E}})$ is a Lie subgroup of $SL(\mathcal{T})$ which acts differentiably from the left on the state manifold Σ. If $\Phi \in g(\hat{\mathcal{E}})$ so $\Phi\colon \mathcal{T} \to \mathcal{T}$ is a symmetry of $\hat{\mathcal{E}}$, and $\sigma \in \Sigma$ is a state we define the left multiplication $\Phi \cdot \sigma \in \Sigma$ to be $\Phi \cdot \sigma = \Phi_\Sigma(\sigma)$. This is well defined as we have seen since Φ_Σ is uniquely determined from the material isomorphism Φ. It allows us to define an equivalence relation (of being in the same orbit of the action) and a set of equivalence classes which is the orbit space, in this case the reduced states of Noll [77] $\Sigma_{red} = \Sigma/g(\hat{\mathcal{E}})$. It has been proved (Dieudonné [22]; p. 60) that if $\{(\sigma_1, \sigma_2) \mid \sigma_1 = \Phi \cdot \sigma_2,\ \Phi \in g(\hat{\mathcal{E}}),\ \sigma_1, \sigma_2 \in \Sigma\}$

is a closed submanifold of $\Sigma \times \Sigma$ then Σ_{red} is a differential manifold and the canonical projection $\Sigma \to \Sigma_{red}$ is a submersion inducing the quotient topology which is consistent with the manifold topology. As our final axiom on the material element we assume that this orbit manifold Σ_{red} exists (i.e. the conditions of the theorem are satisfied). This allows us to differentiably factor out the material equivalence of certain states.

The stabilizer $g(\sigma)$ of a particular state $\sigma \in \Sigma$, namely $g(\sigma) = \{\Phi \in g(\mathcal{E}) \,|\, \Phi \cdot \sigma = \sigma\}$ is a Lie subgroup of $g(\mathcal{E})$ [22, p. 60] which is called the *symmetry group of the state* σ. As in Noll [77], p. 19, $g(\sigma)$ is a Lie subgroup of the orthogonal group $\mathcal{O}(D)$, where D is the deformation defined by the configuration $\hat{G}(\sigma)$. The group $\mathcal{O}(D)$ consists of all linear isomorphisms on T which are isometries with respect to the norm defined by D.

(IV.4) Stress Space Modification.

It is frequently desirable to replace the $\mathrm{Sym}(T^*,T)$ factor in the definition of S by $T^n \times \mathbb{R}^n$. This is particularly helpful in the consideration of adapted frame component systems in space-time, where the stress tensor is to be diagonalized in terms of principal stresses and axes. Thus $S = T^n \times \mathbb{R}^n \times \mathbb{R}^2$ and $R \in S$ is written as $R = (v,\underset{\sim}{\sigma},\eta,\psi)$ where $v = \left(\underset{\sim}{v}_{(1)},\cdots,\underset{\sim}{v}_{(n)}\right)$ is a basis for T and $\underset{\sim}{\sigma} = \left(\sigma_{(1)},\cdots,\sigma_{(n)}\right) \in \mathbb{R}^n$. ($\underset{\sim}{\sigma}$ and $\sigma_{(i)}$ are not to be confused with states σ which are elements of Σ.) If $\sigma \in \Sigma$ and $\hat{G}(\sigma) = G = (D,H,\Theta,g,\dot{U})$ and $\hat{S}(\sigma) = R = (v,\underset{\sim}{\sigma},\eta,\psi)$ then we require that v be orthonormal with respect to D, and the stress $S \in \mathrm{Sym}(T^*,T)$ must be determined from $S(\underset{\sim}{v}^{(i)*}) = \sigma_{(i)}\underset{\sim}{v}_{(i)}$ $1 \le i \le n$ where v^* is the basis of T^* dual to v, $v^* = \left(\underset{\sim}{v}^{(1)*},\cdots,\underset{\sim}{v}^{(n)*}\right)$. Thus, in this case, the deformation D is determined from the stress map $\hat{S}$.

(IV.5) The Heat Flux Function $\hat{H}$.

A number of people have investigated the speed of thermal disturbances in continuous media with the objective in mind of avoiding the paradoxes of Fourier's law of conduction Maugin [58], p. 465 which assumes a direct relationship between thermal gradient g and the heat flux $q^a = \lambda v^a$. See also Müller [74], Carter [10] and Israel[1]. Rather than to attempt to define the velocity of thermal propogation in the context of this general theory, or look at more specific media where kinetic theory and transport processes can be discussed, it is proposed that the heat flux function $\hat{H}$ be defined on the evolution bundle $E\Sigma$ rather than the state space Σ. This allows for an equation of thermal conductivity in which the heat flux depends on the temperature within the past null cone in the infinitesimally recent past.

(IV.6) The Thermodynamic Body Manifold.

Let B be an n-dimensional differential manifold. B is said to be a *Thermodynamic Body Manifold* if B is equipped with a group structure $G(B) = \left(B, \{G_X, X \in B\}, \{(U^B_\alpha, r^B_\alpha), \alpha \in A^B\}, V_B, G_B\right)$, where V_B is an n-dimensional vector space which is given the structure of a thermodynamic material element $\mathcal{E} = (V_B, G, \Sigma, S, E\Sigma, \hat{G}, \hat{S}, \hat{H}, \hat{\rho}, m)$ whose symmetry group $g(\mathcal{E})$ is G_B. Thus we see that the material element $\mathcal{E}$ induces for us the structure of a material element $\mathcal{E}_X$ at each point $X \in B$ and all the $r^B_{\alpha\,X}: B_X \to V_B$ are material isomorphisms. We may write $\mathcal{E}_X = (B_X, G_X, \Sigma_X, S_X, E\Sigma_X, \hat{G}_X, \hat{S}_X, \hat{H}_X, \hat{\rho}_X, m_X)$ where $g(\mathcal{E}_X) = G_X$. The mass element $m \in \overset{n}{\Lambda}(V_B^*)$ defines for us a smooth invariant n-form on B

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[1] Israel, W. Annals of Physics 100, p. 310-331 (1976).

whose value at $X \in B$ is m_X the mass element of $\mathcal{E}_X$. Thus B is orientable, and moreover $G_B = g(\mathcal{E}) \subset SL(V_B)$. The group structure gives us a smooth materially uniform thermodynamic simple body structure on B corresponding to the notions of material uniformity introduced by Noll [78] and Wang [108].

(IV.7) The Motion of a Thermodynamic Body Manifold in a Lorentz Space-Time.

Let M be an n+1-dimensional Lorentz space time. This means that M has a group structure $G(M) = \left(M, \{G_x, x \in M\}, \{(U_\alpha, r_\alpha), \alpha \in A\}, V, G\right)$ and a Lorentz inner product exists on V which is preserved by G. By a Lorentz inner product on V we mean a symmetric bilinear form $I\colon V \times V \to \mathbb{R}$ for which there exists a basis $v = \left(\underset{\sim}{v}_{(1)}, \cdots, \underset{\sim}{v}_{(n)}, \underset{\sim}{v}_{(n+1)}\right)$ of V with $I\left(\underset{\sim}{v}_{(i)}, \underset{\sim}{v}_{(j)}\right) = \eta_{ij} = \mathrm{diag}_{ij}(1,1,\cdots,1,-1)$. A linear isomorphism from V to itself which sends one Lorentz basis (a basis like v satisfying this condition) to another (componentwise in order) is said to be a Lorentz transformation. The Lorentz transformations are those which preserve I, i.e. $I(\underset{\sim}{v}, \underset{\sim}{w}) = I(g\underset{\sim}{v}, g\underset{\sim}{w})$. Thus G is a Lie subgroup of the Lorentz group, and is taken to be the proper orthochronous subgroup. This means our space-time is orientable and time sense preserving. We expect M, or at least the open subset of space-time through which the material medium moves which we have identified with M, to be orientable, because B is, and we can by exterior product with the flow vector, construct an everywhere non-zero n+1-form on M using the non-zero n-form we have on B. Because of the irreversible nature of continuum deformation properties, we need the notion of "future pointing" on time-like vectors to be well defined everywhere and this explains the orthochronous requirement.

An *acceptable motion* of a thermodynamic body manifold B in a

Lorentz space-time M will be determined by a fibration map $P: M \to B$ whose fibres are time-like world lines diffeomorphic to $\mathbb{R}$, and which satisfies certain consistency relationships.

We assume $x \in P^{-1}(X)$, the world line of X so $P(x) = X$. For each $x \in M$ there is associated a state $\sigma_x \in \Sigma_X$. If we parametrize the world line $P^{-1}(X)$ according to proper time τ say $x = \beta(\tau) \in P^{-1}(X)$, $\tau \in \mathbb{R}$ then the curve $\tau \to \sigma_{\beta(\tau)}$ in Σ_X is a smooth curve. (β is defined at least on an open interval in $\mathbb{R}$.) We let $\hat{G}_X(\sigma_x) = (D_x, H_x, \Theta_x, g_x, \dot{U}_x) = C_x$ and $\hat{S}_X(\sigma_x) = (v_x, \underset{\sim}{\sigma}_x, \eta_x, \psi_x)$ where $v_x = \left(\underset{\sim}{v}_{(1)x}, \cdots, \underset{\sim}{v}_{(n)x}\right) \in B_X^n$ and $\underset{\sim}{\sigma}_x = \left(\sigma_{(1)x}, \cdots, \sigma_{(n)x}\right) \in \mathbb{R}^n$. Notice that $\underset{\sim}{\sigma}_x$ and $\sigma_{(i)x}$ are not to be confused with states in Σ_X, just as G_x and G_X are different. Of course $D_x \in \mathrm{Sym}^+(B_X, B_X^*)$ for each $x \in P^{-1}(X)$ and we require $D_x\left(\underset{\sim}{v}_{(i)x}\right) = \underset{\sim}{v}_x^{(i)*}$ where $v_x^* = \left(\underset{\sim}{v}_x^{(1)*}, \cdots, \underset{\sim}{v}_x^{(n)*}\right)$ is the basis of B_X^* dual to v_x. The stress $S_x \in \mathrm{Sym}(B_X^*, B_X)$ is calculated from the relation $S_x\left(\underset{\sim}{v}_x^{(i)*}\right) = \sigma_{(i)x}\underset{\sim}{v}_{(i)x}$ where $\sigma_{(i)x} \in \mathbb{R}$. Thus v_x is the instantaneous metric orthonormal basis which diagonalizes the stress tensor. It may not be unique but does always exist, and we have that the principal axes and stresses are smoothly specified by the generalized stress function $\hat{S}_X$ on Σ_X, as functions of $\tau = \beta^{-1}(x)$.

Of course, we have the flow vector of the material medium which is tangent to the flow lines and of length -1 at each point of M, being also future pointing. In coordinates it is denoted by u^a and the acceleration is $\dot{u}^a = u^a{}_{;b}u^b$. If $M_x^{\perp}$ is the vector subspace of M_x of those vectors orthogonal to u^a in the metric, then the projection $P: M \to B$ induces a natural isomorphism between $M_x^{\perp}$ and B_X for each $x \in P^{-1}(X)$ and each $X \in B$. We insist that $\dot{U}_x \in B_X$ be the image of $\dot{u}^a\big|_x$ under this isomorphism. Also Θ_x, $x \in M$ defines a scalar field on M. We require it to be smooth and to satisfy $\Theta_{,b}\gamma^b_a\big|_x = g_x$ where

we have identified $M_x^{\perp *}$ with B_X^* using the given linear isomorphism (or in this case its dual). Using the same isomorphism on higher tensor spaces as an identification we require $D_x = \gamma_{ab}\big|_x$, $H_x = 2\theta_{abc}\big|_x$, where θ_{abc} was defined earlier, in (I.24), and is orthogonal (to u^d) and symmetric in a and b. If we define $\dot{D}_x = \frac{d}{d\tau} D_{\beta(\tau)}\Big|_{\tau=\beta^{-1}(x)}$ then $\dot{D}_x = 2\theta_{ab}\big|_x$ under the identification if we take convective derivatives. Likewise we can take the proper time derivative of the configuration C_x at x to get $\dot{C}_x = (\dot{D}_x, \dot{H}_x, \dot{\Theta}_x, \dot{g}_x, \ddot{U}_x)$, where $\dot{D}_x = \mathcal{D}\gamma_{ab}\big|_x$ under identification as we have seen, and $\dot{H}_x = 2\mathcal{D}\theta_{abc}\big|_x$, $\dot{\Theta}_x = \Theta_{,a}u^a$, $\dot{g}_x = (\mathcal{D}\Theta_{,b})\gamma^b_a\big|_x$, $\ddot{U}_x = \mathcal{D}\dot{u}^a\big|_x$. Then if under identification we put $q^a\big|_x = \hat{H}_X(\sigma_x, \dot{C}_x)$ then $q^a = \lambda v^a$ in our old notation is the heat flux vector that appears in the energy-momentum tensor.

Likewise, the image of $\hat{S}_X$ when applied to σ_x gives us a basis for B_X namely v_x, which can be transformed through our isomorphism to a basis $r_x = \left(r_{(1)x}, \cdots, r_{(n)x}\right)$ of $M_x^{\perp}$ which is orthonormal in the metric γ_{ab}. This corresponds to our 3 vectors $\underset{\sim}{r}$, $\underset{\sim}{s}$, $\underset{\sim}{t}$ in 3-space (or 4-space-time) and along with the flow vector $\underset{\sim}{u}_x \in M_x$, the (n+1)-frame $\left(r_{(1)x}, \cdots, r_{(n)x}, \underset{\sim}{u}_x\right)$ gives us an orthonormal basis for M_x in the metric g_{ab} at x. This is the adapted frame whose Ricci rotation coefficients we referred to as the coordinate torsion, and used for an explicit formulation of the Einstein equations. Writing $\underset{\sim}{u}_x$ as $\underset{\sim}{r}_{(n+1)x}$ for the moment we have $g_{ab}r^a_{(i)}r^b_{(j)} = \eta_{ij}$ for $i,j = 1,2,\cdots,n+1$.

(IV.8) Thermodynamics: The First and Second Laws.

The first law of thermodynamics is the energy conservation relation $PdV + dU = dQ$ Fermi [28] p. 20 and Truesdell [105] p. 242 where PdV is the work done by the system on the environment, dU is the internal energy gained by the system and dQ is the heat the system picks up

from the environment. The corresponding equation in relativity is, as we have seen in (III.5),

$$(T^{ab}u_b)_{;a} = \underbrace{\hat{T}^{ab}\theta_{ab} - \lambda v^{a}\dot{u}_{a}}_{\substack{\text{Energy flow into system}\\ \text{per unit 4-volume due}\\ \text{to work done on the}\\ \text{system by stresses and}\\ \text{heat flux (corresponds}\\ \text{to } -PdV\text{)}}} = \underbrace{(\epsilon u^b)_{;b}}_{\substack{\text{Internal energy}\\ \text{gained by the}\\ \text{system per unit}\\ \text{4-volume (i.e.}\\ \text{per unit 3-vol-}\\ \text{ume per unit}\\ \text{time) (corres-}\\ \text{ponds to } dU\text{)}}} + \underbrace{(\lambda v^b)_{;b}}_{\substack{\text{Heat flow out of}\\ \text{the system to the}\\ \text{environment per}\\ \text{unit 4-volume}\\ \text{(corresponds to}\\ -dQ\text{)}}} ,$$

where we have divided the differential form of the equation by the infinitesimal differential of 4-volume to obtain this finite form.

Ehlers [25] p. 822, Day [21] p. 77, Truesdell [105] p. 245, Maugin [58] p. 471 and others have treated entropy in terms of a vector called entropy flux which needs some clarification here in this context because of diverse thermodynamic formulations (Müller [73] p. 260). It is convenient to treat the concept of entropy and the second law in the following manner. This is a refined technique of the one used in classical thermodynamics of Fermi [28] where we pick out the TdS term from the heat flow in the first law.

For our infinitesimally small 3-dimensional space of material points moving along a world line in space-time in the rest frame we have:

ϵ = energy density per unit 3-volume,

$(\epsilon u^a)_{;a}$ = internal energy gained by the system per unit 3-volume per unit time (= per unit 4-volume),

ρ = mass density in rest frame (not including mass equivalent of thermal energy),

$(\rho u^a)_{;a} = 0$ = mass gained by the system per unit 4-volume,

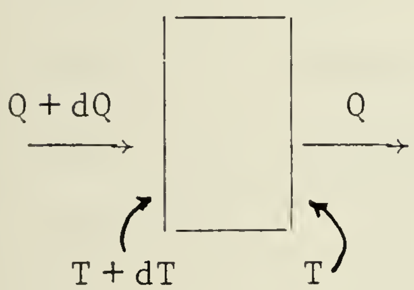
η = entropy density (per unit 3-volume in rest frame),

$(\eta u^a)_{;a}$ = *total* entropy gained by the system per unit 4-volume.

We let $\dot{\eta}_{int}$ be the entropy 3-density gained by the system internally as a result of irreversible processes, per unit time. It has units of entropy per unit 4-volume. Then

$$(\eta u^a)_{;a} = \dot{\eta}_{int} + \frac{\Delta Q}{\Theta} = \dot{\eta}_{int} - \left(\frac{\lambda v^a}{\Theta}\right)_{;a} ,$$

where ΔQ is the heat gained by the system per unit 4-volume at temperature Θ. We might ask why is Θ included in the differentiation and we put $\left(\frac{\lambda v^a}{\Theta}\right)_{;a}$ instead of $\left(\frac{(\lambda v^a)_{;a}}{\Theta}\right)$?



To see this we imagine a system where heat $Q + dQ$ flows into a system at temperature $T + dT$ and Q flows out at temperature T, so the entropy change due to heat flow is $dS = d\left(\frac{Q}{T}\right) = \frac{dQ}{T} - Q\frac{dT}{T^2}$ and not simply $\frac{dQ}{T}$.

For our space-time case we may write $\dot{\eta}_{int} = S^a{}_{;a}$ where $S^a = \eta u^a + \frac{\lambda v^a}{\Theta}$ is called the *entropy flux*, and $\dot{\eta}_{int}$ is the *entropy (density) creation rate*. We have $\dot{\eta}_{int} \geq 0$ always (the second law) and $\dot{\eta}_{int} = 0$ for reversible processes.

The *free energy* ψ is again a 3-density in rest frame scalar function defined by $\psi = \epsilon - \eta\Theta$ Fermi [28] p. 78, Truesdell [105] p. 245. It is also called the Helmholtz free energy by chemists who use the symbol F for it, in contrast to the Gibbs free energy. We may write, differentiating and substituting from the first law,

$$\Theta\dot{\eta}_{int} = -(\psi u^a)_{;a} - \eta\dot{\Theta} + \hat{T}^{ab}\theta_{ab} - \lambda v^a\dot{u}_a - \lambda v^a(\ln\Theta)_{,a}$$

$$= -\rho(\dot{\bar{\psi}} + \bar{\eta}\dot{\Theta}) + \hat{T}^{ab}\theta_{ab} - \lambda v^a\dot{u}_a - \lambda v^a(\ln\Theta)_{,a} \geq 0,$$

where "$\cdot$" is proper time derivative, and $\eta = \rho\bar{\eta}$, $\psi = \rho\bar{\psi}$ and we call $\bar{\psi}$ the *specific free energy* and $\bar{\eta}$ the *specific entropy*. This gives us the Clausius-Duhem form inequality Maugin [58] p. 471, eqn. (25) including the acceleration $(\dot{u}_a)$ term which he does not have. See also Noll [77] p. 49.

(IV.9) Axiom of Thermodynamic Reversibility (A.T.R.).

It is possible to define reversibility for elements in the evolution space that specify for us infinitesimal configuration change rates. If $(\sigma, v_{\hat{G}(\sigma)}) \in E\Sigma$, i.e. $\sigma \in \Sigma$ and $v_{\hat{G}(\sigma)}$ is a consistent vector in the tangent space to G at the configuration $\hat{G}(\sigma)$, then we say $(\sigma, v_{\hat{G}(\sigma)})$ is *reversible* if the following condition holds. There exists another consistent vector $v'_{\hat{G}(\sigma)}$ at $\hat{G}(\sigma)$ such that $\hat{\rho}(\sigma, v_{\hat{G}(\sigma)}) = -\hat{\rho}(\sigma, v'_{\hat{G}(\sigma)})$ where "$-$" refers to scalar multiplication by -1 in the linear vector space Σ_σ the tangent space to Σ at σ. We say $(\sigma, v'_{\hat{G}(\sigma)})$ is *opposite* to $(\sigma, v_{\hat{G}(\sigma)})$. Since $\hat{G}_*$ is linear on the fibres we see that $v'_{\hat{G}(\sigma)} = -v_{\hat{G}(\sigma)}$ is necessary but not sufficient for $(\sigma, v'_{\hat{G}(\sigma)})$ to be opposite to $(\sigma, v_{\hat{G}(\sigma)})$, and this condition is sufficient if and only if $(\sigma, v_{\hat{G}(\sigma)})$ is reversible. We observe that ρ need not be linear on the fibres. If an opposite evolution element exists it must of course be unique.

Let $\sigma \in \Sigma$ be a state and let $P: [0, d_p] \to G$ be a smooth process with $P(0) = G = \hat{G}(\sigma)$, and let $Q: [0, d_p] \to \Sigma$ be the lifted state process with $Q(0) = \sigma$. We say that P is a *reversible process* if there exists a process $\bar{P}: [0, d_{\bar{p}}] \to G$ with $\bar{P}(0) = P(d_p)$ such that for the lifted process $\bar{Q}: [0, d_{\bar{p}}] \to \Sigma$ with $\bar{Q}(0) = Q(d_p)$ we have $\bar{Q}(d_{\bar{p}}) = \sigma$. Hence also $\bar{P}(d_{\bar{p}}) = G$. P is said to be *completely reversible* if $\frac{d}{dt} P(t) = P'(t)$ satisfies the condition that $\big(Q(t), P'(t)\big) \in E\Sigma$

is reversible for all t. For a more general process P we may find P is not differentiable at some $t_0 \in [0,d_p]$ which can happen for finitely many points in the interval. Here we require the evolution element $(Q(t_0),P'(t_0))$ to be reversible when $P'(t_0)$ is replaced by either the right hand or left hand derivative at t_0. It is easy to see that a completely reversible process is reversible, simply by taking $d_{\overline{P}} = d_P$ and $\overline{P}(t) = P(d_p - t)$.

The second law, when expressed in the form analogous to the Clausius-Duhem inequality, can be characterized strictly in terms of the evolution element $(\sigma, v_{\hat{G}(\sigma)})$ under consideration. Interpreting $v_{\hat{G}(\sigma)} = \dot{C}$ as the (proper) time derivative of the configuration, we can write the second law down in terms of intrinsic (body manifold) quantities as

$$\dot{\eta}_{int} = -\frac{\psi\theta}{\Theta} - \frac{\dot{\psi}}{\Theta} - \eta\frac{\dot{\Theta}}{\Theta} + \frac{\hat{T}^{ab}\theta_{ab}}{\Theta} - \frac{\lambda v^a \dot{u}_a}{\Theta} - \frac{\lambda v^a \Theta_{,a}}{\Theta^2} \geq 0$$

$$= -\frac{\psi}{\Theta}\left(\frac{\mathrm{tr}\,\dot{D}}{2}\right) - \frac{\dot{\psi}}{\Theta} - \frac{\eta\dot{\Theta}}{\Theta} + \frac{\mathrm{tr}(\dot{D}\circ S)}{2\Theta} - \frac{\langle D(\dot{U}),\hat{H}(\sigma,v_{\hat{G}(\sigma)})\rangle}{\Theta} - \frac{\langle g,\hat{H}(\sigma,v_{\hat{G}(\sigma)})\rangle}{\Theta^2}$$

where $D \in \mathrm{Sym}^+(\mathcal{T},\mathcal{T}^*)$, $S \in \mathrm{Sym}(\mathcal{T}^*,\mathcal{T})$ are deformation and stress, η and ψ are entropy and free energy density respectively associated with σ, $g \in \mathcal{T}^*$ is body thermal gradient, $\dot{U} \in \mathcal{T}$ the acceleration, and "•" indicates the time derivatives of these quantities determined by $v_{\hat{G}(\sigma)}$. Observe that $\theta_{ab}\hat{T}^{ab} = \frac{1}{2}\,\mathrm{tr}(\dot{D}\circ S) = \frac{1}{2}\,\mathrm{tr}(S\circ\dot{D})$ and $\theta = \frac{\mathrm{tr}\,\dot{D}}{2} \equiv \frac{1}{2}\,\mathrm{tr}(\dot{D}\circ D^{-1}) = \frac{1}{2}\,\mathrm{tr}(D^{-1}\circ\dot{D})$ defines $\mathrm{tr}\,\dot{D}$. Thus for any element $(\sigma,v_{\hat{G}(\sigma)}) \in E\Sigma$ we can calculate $\dot{\eta}_{int}$ associated with it. We now state two more axioms for our material element.

<u>Second Law of Thermodynamics</u>. *For any* $(\sigma,v_{\hat{G}(\sigma)}) \in E\Sigma$ *we have* $\dot{\eta}_{int}(\sigma,v_{\hat{G}(\sigma)}) \geq 0$.

<u>Axiom of Thermodynamic Reversibility</u> (A.T.R.). *The element* $(\sigma, v_{\hat{G}(\sigma)})$ *is reversible if and only if* $\dot{\eta}_{int}(\sigma, v_{\hat{G}(\sigma)}) = 0$.

The latter axiom needs some clarification here. First let us look at the energy conservation relation (the first law). If the evolution element $(\sigma, v_G) \in E\Sigma$, $G = \hat{G}(\sigma)$ is specified, then everything in this equation is determined except $q^b{}_{;b} = (\lambda v^b)_{;b}$ by (σ, v_G) and we may write the first law determining $q^b{}_{;b}$ as

$$(\lambda v^b)_{;b} = \theta_{ab}\hat{T}^{ab} - (\epsilon u^b)_{;b} - \lambda v^a \dot{u}_a$$
$$= \frac{1}{2}\operatorname{tr}(\dot{D} \circ S) - (\psi + \eta\Theta)\theta - \dot{\psi} - \dot{\eta}\Theta - \eta\dot{\Theta} - \langle \hat{H}(\sigma, v_G), D(\dot{U}) \rangle$$

where we have written the result in intrinsic (body) form. It is worth noting that $\langle \hat{H}(\sigma, v_G), D(\dot{U}) \rangle = \langle D \circ \hat{H}(\sigma, v_G), \dot{U} \rangle$. In words, the evolution element determines the work done on the system, the internal energy gained by the system and therefore (by energy conservation) the heat flow into the system from the environment.

Heuristically then, let us understand the principle behind the axiom of thermodynamic reversibility. It is easy to see that if $(\sigma, v_{\hat{G}(\sigma)})$ is reversible, then $\dot{\eta}_{int}$ for this element is zero, for $\hat{\rho}(\sigma, v_{\hat{G}(\sigma)}) = -\hat{\rho}(\sigma, -v_{\hat{G}(\sigma)}) \Rightarrow \dot{\eta}_{int}(\sigma, v_{\hat{G}(\sigma)}) = -\dot{\eta}_{int}(\sigma, -v_{\hat{G}(\sigma)})$ which means it is zero by the second law which states it is always ≥ 0 for any evolution element. Physically, corresponding to $(\sigma, v_G) \in E\Sigma$, $G = \hat{G}(\sigma)$, in a time dt which is infinitesimal, our configuration $C = \hat{G}(\sigma) = G$ changes to $C + dC = C + \dot{C}dt = C + v_G dt$, since G is a submanifold of a vector space. (Notice that $\mathrm{Sym}^+(T, T^*) \subset \mathrm{Sym}(T, T^*)$). We can then, in the following time interval of equal length dt reverse the direction of the process to return us to the original configuration. If we have returned to the original state to within first order, our

evolution element was reversible. If not, our state has changed, the rate of which is $\hat{\rho}(\sigma,v_G) + \hat{\rho}(\sigma,-v_G)$. If we return to the original state in this two step infinitesimal process, there is no change of internal energy, nor can there by any net work done on the system (to first order). Hence there is no net heat gained or lost by the system, which means that if $\dot{\eta}_{int}(\sigma,v_G) > 0$ we have greater entropy in the final than in the initial state, since $\dot{\eta}_{int}(\sigma,-v_G) \geq 0$. This is impossible, since η is a state function, so $\dot{\eta}_{int}(\sigma,v_G) = 0$ by physical arguments.

It is the converse of this which states that thermodynamic reversibility implies rheological reversibility which is new, and characterizes thermodynamic material media.

(IV.10) Thermolinear Materials.

In most cases we are interested in special additional requirements being imposed on the material element $\mathcal{E}$. One of the most common is to assume that every $\tilde{G}_G$ is a vector subspace of G_G for $G \in G$, and furthermore that $E\Sigma$ is a vector bundle (Dieudonné [22] p. 105), the base space of course being the state manifold Σ. We say $\mathcal{E}$ is a *thermolinear element* if $E\Sigma$ is a vector bundle, and if $\hat{\rho}_\sigma = \hat{\rho}(\sigma,\cdot)\colon \tilde{G}_{\hat{G}(\sigma)} \to \Sigma_\sigma$ is a linear mapping on the vector spaces. If we follow Noll's approach of letting Π be the collection of all admissible processes, then every such process for a thermolinear material element will be completely reversible. Thus $0 = \dot{\eta}_{int}(\sigma,v_G)$, $G = \hat{G}(\sigma)$ for all $(\sigma,v_G) \in E\Sigma$. We say $\mathcal{E}$ is *thermoelastic* (Day [21] p. 18) if $E\Sigma$ is a vector bundle, and $\hat{G}\colon \Sigma \to G$ is a diffeomorphism. Thus a (thermo) elastic material has no memory and the state is determined by the configuration. This is similar to the notion of a

perfect material, Lianis [49] p. 302. Clearly, every thermoelastic material is thermolinear, and therefore all its evolution elements are reversible.

We say $\mathcal{E}$ is a *fluid* if $g(\mathcal{E}) = SL(T)$, and $\mathcal{E}$ is a *solid* if $g(\mathcal{E}) \subset o(I)$, the orthogonal group of I where $I \in Sym^+(T,T^*)$, for some deformation I. As Noll [77] p. 19 has done, we can define isotropic states and isotropic material elements. A state $\sigma \in \Sigma$ is *isotropic* if its symmetry group is the orthogonal group of its deformation, i.e. $g(\sigma) = so(D)$ where D is the deformation determined from the configuration $\hat{G}(\sigma)$. A material element $\mathcal{E}$ is *isotropic* if it has at least one isotropic state. An isotropic state has the largest possible symmetry group since we know that for any state σ, $g(\sigma) \subset so(D) \cap g(\mathcal{E})$.

<u>Propositions</u>: *Let* $\mathcal{E}$ *be a material element with* $g(\mathcal{E}) \neq SL(T)$. *Then* $g(\mathcal{E}) = g(\sigma)$ *if and only if* σ *is isotropic.*

Let $\sigma \in \Sigma$ *be a state,* $S \in Sym(T^*,T)$, $D \in Sym^+(T,T^*)$ *be the stress and deformation respectively, associated with* σ. *Then* $\forall\ \Phi \in g(\sigma)$ *we have* $\Phi \circ S \circ D = S \circ D \circ \Phi$. *Thus if* σ *is an isotropic state,* $S = -PD^{-1}$ *where* $P \in \mathbb{R}$ *is called the pressure.*

If $\sigma \in \Sigma$ *and* $\Phi \in g(\mathcal{E})$ *then* $g(\Phi \cdot \sigma) = \Phi \circ g(\sigma) \circ \Phi^{-1}$.

<u>Proofs</u>: See Noll [77] p. 19, and follow his approach here.

An isotropic material element which is a solid is called an *isotropic solid*. For such a material element $\mathcal{E}$ there exists a deformation I with $g(\mathcal{E}) = so(I)$. The converse is clearly also true, namely $\exists\ I \in Sym^+(T,T^*)$ with $g(\mathcal{E}) = so(I)$ implies that our isotropic material is a solid.

A material which is not a solid is called a *fluid crystal*. If $\mathcal{E}$ is a solid and $g(\mathcal{E}) \subset o(I)$, where I of course need not be unique, then I induces an invariant Riemannian metric on the body manifold B whose material element is $\mathcal{E}$. Wang [108] p. 83 has shown that if $\mathcal{E}$ is an isotropic solid, then for any I with $g(\mathcal{E}) = so(I)$, the Christoffel symbols of the induced Riemannian metric are the components of a material connection (a connection on the group structure of the Body manifold). For a thermoelastic material $\mathcal{E}$ the function $\hat{\rho}$ is determined uniquely in a trivial way (if we identify G and Σ) and $\hat{S}$ carries the physical information.

To explain the notion of a perfect material (of which there are variations in the literature) Lianis [49] p. 302, Souriau [99] p. 345 and Lodge [55] p. 145, (Oldroyd's form) in this general setting we introduce the following definitions. We say that $\mathcal{E}$ is *symmetry-thermoperfect* (or s-*thermoperfect* for short) if $\forall\, \sigma \in \Sigma$, $\Phi \in g(\mathcal{E})$, $\Phi \circ S_\sigma \circ D_\sigma = S_\sigma \circ D_\sigma \circ \Phi$ where $S_\sigma \in \mathrm{Sym}(T^*, T)$, $D_\sigma \in \mathrm{Sym}^+(T, T^*)$ are the stress and deformation respectively associated with σ. Also, $\mathcal{E}$ is *thermoperfect* if it is s-thermoperfect and thermolinear. (We shall see that thermolinear corresponds to Oldroyd's form of a perfect material.) We say $\mathcal{E}$ is *elastoperfect* if it is thermoperfect and thermoelastic. A *thermoperfect fluid* is a thermoperfect material which is a fluid. Likewise we define an s-*thermoperfect fluid* and *elastoperfect fluid*, by combining the definitions. It is easy to see that for an s-thermoperfect fluid, $S_\sigma = -P(\sigma)D_\sigma^{-1}$ where $P\colon \Sigma \to \mathbb{R}$ is a smooth function. For an elastoperfect fluid P is a function of configuration and there is no hereditary or history dependence.

Let $\mathcal{E}$ be a thermolinear material element, $P \in \Pi$ a process with duration d_p and $P(0) = G = \hat{G}(\sigma)$ for some $\sigma \in \Sigma$. Let $Q\colon [0, d_p] \to \Sigma$

be the lifted process with $Q(0) = \sigma$. Let $f: [0,d_p] \to [0,d_{\bar{p}}]$ be a smooth monotone increasing function with derivative strictly positive everywhere satisfying $f(0) = 0$, $f(d_p) = d_{\bar{p}}$. Define $\bar{P}: [0,d_{\bar{p}}] \to G$ by $\bar{P} = P \circ f^{-1}$ and $\bar{Q}: [0,d_{\bar{p}}] \to \Sigma$ by $\bar{Q} = Q \circ f^{-1}$. Then we can see that $\bar{P} \in \Pi$ is an admissible process, and $\bar{Q}$ is the unique lifted state process corresponding to $\bar{P}$ and satisfying $\bar{Q}(0) = \sigma$. In other words, this means that for a thermolinear material element the final state depends on the initial state and the path of the process in the configuration space but not on the time parametrization. Alternatively, in deforming from configuration C_1 to configuration C_2 along a prescribed path, the result physically does not depend on whether the deformation was slow or fast. The proof of this is immediate from the definition of thermolinearity. Hence we can say that the stress responds immediately to the configuration change [55, p. 145].

We can also see that an s-thermoperfect isotropic solid has $S_\sigma = -P(\sigma)D_\sigma^{-1}$, since $\mathcal{g}(\mathcal{E})$ is a special orthogonal group. Let $\mathcal{E}$ be s-thermoperfect. Now $\mathcal{g}(\mathcal{E})$ acts differentiably from the left on Σ with orbit manifold Σ_{red}, and $\Phi \in \mathcal{g}(\sigma) \iff \Phi \cdot \sigma = \sigma$ so $\mathcal{g}(\sigma)$ is the stabilizer of σ, a Lie subgroup of $\mathcal{g}(\mathcal{E})$. Let $\Phi \in \mathcal{g}(\mathcal{E})$ and $\sigma' = \Phi \cdot \sigma$ so

$$
\begin{aligned}
S_{\sigma'} \circ D_{\sigma'} &= \Phi \circ S_\sigma \circ \Phi^* \circ \Phi^{*-1} \circ D_\sigma \circ \Phi^{-1} \\
&= \Phi \circ S_\sigma \circ D_\sigma \circ \Phi^{-1} = S_\sigma \circ D_\sigma .
\end{aligned}
$$

Thus $S_\sigma \circ D_\sigma \in \mathrm{Lin}(\mathcal{T})$ is a constant on the orbits and is therefore a well defined function on Σ_{red} which is smooth (Dieudonné [22] p. 63). In the equation $S_\sigma \circ D_\sigma \circ \Phi = \Phi \circ S_\sigma \circ D_\sigma$, $\sigma \in \Sigma$, $\Phi \in \mathcal{g}(\mathcal{E})$ if we drop the σ and write $S \circ D \circ \Phi = \Phi \circ S \circ D$, and use $D: \mathcal{T} \to \mathcal{T}^*$ as a linear isomorphism identifying $\mathcal{T}$ with $\mathcal{T}^*$ (i.e. a metric or norm) then using $*$ to represent both dual maps and adjoints with respect to D we have $\Phi^* \circ D^* \circ S^* = D^* \circ S^* \circ \Phi^*$ or, since $D^* = D^{-1} = D$ and $S^* = S$,

$\Phi^* \circ D \circ S = D \circ S \circ \Phi^*$. In this form, we can drop the identification of $\mathcal{T}$ and $\mathcal{T}^*$ using D and so we see that if $\sigma' = \Phi \cdot \sigma$, $D_{\sigma'} \circ S_{\sigma'} = \Phi^{*-1} \circ D_\sigma \circ S_\sigma \circ \Phi^* = D_\sigma \circ S_\sigma$. Therefore, also $D_\sigma \circ S_\sigma \in \mathrm{Lin}(\mathcal{T}^*)$ is constant on the orbits and is hence a well defined smooth function on Σ_{red}. We see that the condition $S_\sigma \circ D_\sigma$ = constant on orbits, is sufficient also for $\mathcal{E}$ to be s-thermoperfect, as is the condition $D_\sigma \circ S_\sigma$ = constant on orbits.

(IV.11) Electromagnetic Theory in Relativity

We should like to include electromagnetic effects in this theory specifying just what parts of the field or its derivatives are determined by the state or evolution element, and how Maxwell's equations aid us in determining the relations satisfied. For a general relativistic Lorentzian space-time we represent the electromagnetic field including induction effects[1] by two skew-symmetric tensors F_{ab} and G_{ab}. We call F_{ab} the *electric field-magnetic induction tensor* and G_{ab} the *electric induction-magnetic field tensor*. The corresponding dual tensors are $\mathcal{F}^{de} = \frac{1}{2}\,\varepsilon^{deab}F_{ab}$ and $\mathcal{G}^{de} = \frac{1}{2}\,\varepsilon^{deab}G_{ab}$ with inversion formulas $F^{ab} = -\frac{1}{2}\,\varepsilon^{abde}\mathcal{F}_{de}$ and $G^{ab} = -\frac{1}{2}\,\varepsilon^{abde}\mathcal{G}_{de}$ where the metric g_{ab} is used to raise and lower indices. We can express these tensors in terms of two space-like vectors orthogonal to the flow in the matter flow frame in each case. We then have (using $\varepsilon_{abc} = \varepsilon_{abcd}u^d$, $\varepsilon_{abcd} = \sqrt{g}\,\epsilon_{abcd}$) $F_{ab} = -\varepsilon_{abc}B^c - E_a u_b + E_b u_a$ and $G_{ab} = -\varepsilon_{abc}H^c - D_a u_b + D_b u_a$ so that the duals are $\mathcal{F}_{de} = -\varepsilon_{dea}E^a + B_d u_e - B_e u_d$ and $\mathcal{G}_{de} = -\varepsilon_{dea}D^a + H_d u_e - H_e u_d$. We call E^a the *electric field*, B^a the *magnetic induction*, H^a the *magnetic field* and D^a the *electric induction*. These are vector fields

[1] Lichnerowicz, A., Relativistic Hydrodynamics and Magnetohydrodynamics, W.A. Benjamin, Inc. N.Y. 1967, p. 84.

orthogonal to the flow u^a. As vectors in $M_x^{\perp}$ we write them as $\vec{E}$, $\vec{B}$, $\vec{H}$ and $\vec{D}$.

The induction vectors are given in terms of the field vectors by the relations $\vec{B} = \mu\vec{H}$ and $\vec{D} = \chi\vec{E}$. We call μ the *magnetic permeability* and χ the *electric permitivity* or *susceptibility*, χ being directly related to the dielectric constant. For isotropic media these are scalar functions of the state $\sigma \in \Sigma$ and in the non-isotropic case they are positive definite symmetric matrix functions of σ in a particular basis.

Maxwell's equations are $dF_{ab|c} = 0$ (equivalently $F^{ab}{}_{;b} = 0$) and $G^{ab}{}_{;b} = J^a$ (equivalently $dG_{ab|c} = -\varepsilon_{abcd}J^d$). Since $G^{ab} = -G^{ba}$, $J^a{}_{;a} = 0$ which is conservation of charge, J^a being the charge flow vector for electric charge. We can write $J^a = j^a + \rho_{(c)}u^a$ where $j^a u_a = 0$. The *charge density* in the matter rest frame is $\rho_{(c)}$ and the vector j^a is the *current density*.

The contribution to the energy momentum tensor due to electromagnetic effects is given by $T_{(em)}{}^{ab} = F^{ac}G_c{}^b + \frac{1}{4} g^{ab}F^{cd}G_{cd}$. In terms of the fields we have

$$T_{(em)}{}^{ab} = H^aB^b - \gamma^{ab}\,\vec{B}\cdot\vec{H} - (\vec{D}\times\vec{B})^a\,u^b + E^aD^b$$
$$- u^a(\vec{E}\times\vec{H})^b - (\vec{E}\cdot\vec{D})u^au^b + \frac{1}{2}\,g^{ab}(\vec{B}\cdot\vec{H} - \vec{E}\cdot\vec{D}),$$

where $(\vec{D}\times\vec{B})^a = -\varepsilon^{acd}D_cB_d$. Thus $T_{(em)}{}^{ab}$ is not in general symmetric, but may be in specific cases, such as the unidirectional space-times where $\vec{E}$, $\vec{D}$, $\vec{B}$ and $\vec{H}$ are all parallel. If $T_{(em)}{}^{ab}$ is not symmetric then the electromagnetic effects by and of themselves are not conserving angular momentum, but when coupled to the material contributions the total energy momentum tensor is symmetric and represents angular momentum conservation. We have $T^{ab} = T_{(em)}{}^{ab} + T_{(mat)}{}^{ab}$ with an intricate inter-

relation between material stress and electromagnetic field. Here $T^{ab} = T^{ba}$ and $T^{ab}{}_{;b} = 0$. In the case $\mu = \chi = 1$ so $G_{ab} = F_{ab}$ we have $T_{(em)}{}^{ab}{}_{;b} = F^{ad}J_d$, and $T_{(em)}{}^{ab}$ is symmetric. This holds in particular for the vacuum case where $T^{ab} = T_{(em)}{}^{ab}$ and $J_a = 0$. It is the total energy momentum tensor T^{ab} that will always be the one of interest, with orthogonal parts determined by the constitutive theory. Henceforth we will refer no more to $T_{(em)}{}^{ab}$.

Let us see how to represent first order derivatives of fields such as H^a or E^a on the body manifold. Clearly $H^a = P^a_\alpha H^\alpha$ and $E^\alpha = P^\alpha_a E^a$, $E_\alpha = P^a_\alpha E_a$, $E_a = P^\alpha_a E_\alpha$ where the α index is raised and lowered using $g_{\alpha\beta}(\tau)$. Similar equations hold for D^α and B_α. Now using $0 = P^\alpha_{a\wedge b} = P^\alpha_{a,b} + P^\beta_a \Gamma_\beta{}^\alpha{}_\gamma P^\gamma_b - P^\alpha_c \hat{\Gamma}_a{}^c{}_b$ and $\Gamma_\beta{}^\alpha{}_\gamma = \{{}_\beta{}^\alpha{}_\gamma\}(\tau) - K_\beta{}^\alpha{}_\gamma(\tau)$, $\hat{\Delta}_a{}^d{}_b = P^d_\alpha P^\beta_a P^\gamma_b K_\beta{}^\alpha{}_\gamma(\tau)$, $\hat{\Gamma}_a{}^c{}_b = \{{}_a{}^c{}_b\} - \hat{K}_a{}^c{}_b$ and $\gamma^d_c \hat{K}_a{}^c{}_b = \hat{\Delta}_a{}^d{}_b - u^d{}_{;b}u_a - u^d{}_{;a}u_b - \dot{u}^d u_a u_b$ which are results from (I.22), (I.27), we obtain

$$0 = P^\alpha_{a,b} + P^\beta_a\{{}_\beta{}^\alpha{}_\gamma\}(\tau)P^\gamma_b - P^\alpha_c\left(\{{}_a{}^c{}_b\} + u^c{}_{;b}u_a + u^c{}_{;a}u_b + \dot{u}^c u_a u_b\right) . \tag{IV.11.1}$$

Now, $H^\alpha = P^\alpha_a H^a$ so that $H^\alpha{}_{,\beta} = (P^\alpha_a H^a)_{,b} P^b_\beta = P^\alpha_{a,b} P^b_\beta H^a + P^\alpha_a P^b_\beta H^a{}_{,b}$. We define the instantaneous proper time covariant derivative (using Christoffel symbols) on the body manifold as $H^\alpha{}_{;\beta} = H^\alpha{}_{,\beta} + H^\gamma\{{}_\gamma{}^\alpha{}_\beta\}(\tau)$, and of course also $H^a{}_{;b} = H^a{}_{,b} + H^c\{{}_c{}^a{}_b\}$ on space-time. Then multiplying the equation (IV.11.1) which we obtained earlier by $H^a P^b_\beta$ and simplifying we find that $H^\alpha{}_{;\beta} = P^\alpha_a P^b_\beta H^a{}_{;b}$ which is the covariant derivative projection formula for first Christoffel symbol derivative on the body manifold. From the definition of $\{{}_\beta{}^\alpha{}_\gamma\}(\tau)$ in terms of $g_{\alpha\beta}(\tau)$ we see that $g_{\alpha\beta;\gamma} = 0$ so $H_{\alpha;\beta} = P^a_\alpha P^b_\beta H_{a;b}$. We have similar projections for $E^a{}_{;b}$, $E_{a;b}$, $D_{a;b}$, $B^a{}_{;b}$ and so forth. Now $g_{\alpha\beta;\gamma} = 0$ for each τ, so using the

material connection on B we have,

$$g_{\alpha\beta|\gamma} = g_{\alpha\beta;\gamma} + g_{\delta\beta}K_{\alpha\ \gamma}^{\ \delta}(\tau) + g_{\alpha\delta}K_{\beta\ \gamma}^{\ \delta}(\tau) = 2K_{(\alpha\beta)\gamma}(\tau).$$

Lifting to space-time we find the orthogonal tensor θ_{abc} which is determined by the configuration and hence the state of the material element at (X,τ) gives us all the information we need to obtain the orthogonal derivatives of $g_{\alpha\beta}(\tau)$ on the body. In fact $\theta_{abc} = P^{\alpha}_{a}P^{\beta}_{b}P^{\gamma}_{c}K_{(\alpha\beta)\gamma} = \frac{1}{2} P^{\alpha}_{a}P^{\beta}_{b}P^{\gamma}_{c}g_{\alpha\beta|\gamma}$. In our original configuration notation $H_x = g_{\alpha\beta|\gamma}(X,\tau)$.

(IV.12) Dependence on the Choice of Material Connection

Of course there may be more than one material connection on the body manifold. If $\Gamma_{\beta\ \gamma}^{\ \alpha}$ and $\Gamma'^{\ \alpha}_{\beta\ \gamma}$ are both material connections on B it is easy to see that $a^{\alpha}_{\ \beta}\big|_X = (\Gamma'^{\ \alpha}_{\beta\ \gamma} - \Gamma_{\beta\ \gamma}^{\ \alpha})t^{\gamma}\big|_X \in \mathcal{g}_X$ the Lie algebra at X where we view $a^{\alpha}_{\ \beta}\big|_X\colon B_X \to B_X$ as a linear map, and where the vector t^{γ} is arbitrary. This is simply the statement that parallel transports preserve material isomorphisms. Hence writing $\theta_{\alpha\beta\gamma} = K_{(\alpha\beta)\gamma}(\tau)$ we have $-a_{(\alpha\beta)} = \Delta\theta_{\alpha\beta\gamma}t^{\gamma} = (\theta'_{\alpha\beta\gamma} - \theta_{\alpha\beta\gamma})t^{\gamma}$ where $a^{\alpha}_{\ \beta}\big|_X \in \mathcal{g}_X$ and $g_{\alpha\beta}(\tau)$ is used to raise and lower indices. Since $H_X = 2\theta_{\alpha\beta\gamma}\big|_X$ is the basic spatial gradient of deformation the transformation $H_X \to H_X + 2\Delta\theta_{\alpha\beta\gamma}\big|_X$ ought to be trivial in a certain sense when considering the state space, stress and evolution map of the material element at X. If $G = \{e\}$ is the trivial group, the material connection on the body is unique and this provides no constraint. The larger the group G, the less able we are to measure spatial gradients of deformation in any canonical way. If G is a subgroup of $\mathrm{Unim}(V)$, then $g^{\alpha\beta}(\tau)\Delta\theta_{\alpha\beta\gamma} = 0$, for instance. On the other hand, if a particular material connection on B is singled out for dynamical purposes, we can find spatial gradient of deformation without

trouble, and need not worry about the dependence of the dynamical properties of the state on particular variations in H related to a change in material connection on B.

(IV.13) The Maxwell Equations

It is straightforward to verify that Maxwell's equations on space-time can be written in the following form:

$$-\rho_{(c)} = 2\nu^c H_c + \dot{u}_a D^a - D^b{}_{;b} ,$$

$$j^e = \varepsilon^{ecd}(H_{c;d} + H_c \dot{u}_d) - D^e{}_{;b} u^b + u^e \dot{u}_a D^a - D^e \theta + u^e{}_{;b} D^b ,$$

$$0 = 2\nu^a E_a - \dot{u}_d B^d + B^e{}_{;e} ,$$

$$0 = \varepsilon^{bac}(E_{a;c} + E_a \dot{u}_c) + B^b{}_{;e} u^e - u^b \dot{u}_d B^d + B^b \theta - u^b{}_{;e} B^e .$$

These are to be compared with the usual classical equations $\operatorname{div} \vec{D} = \rho_{(c)}$, $\vec{J} = \operatorname{curl} \vec{H} - \frac{\partial \vec{D}}{\partial t}$, $\operatorname{div} \vec{B} = 0$ and $\operatorname{curl} \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$. We see additional acceleration expansion and rotation terms, some of which are obvious contorsion type terms that can be incorporated by a change of connection i.e. try $\overset{\Delta}{K}{}_c{}^e{}_d = \gamma^e_c \dot{u}_d + u^e u_c \dot{u}_d$.

Let us consider the appearance of Maxwell's equations on the body manifold. First of all observe that

$$E^a{}_{;b} = (\gamma^a_c - u^a u_c) E^c{}_{;d} (\gamma^d_b - u^d u_b)$$

$$= P^a_\alpha P^\beta_b E^\alpha{}_{;\beta} + u^a u_{c:b} E^c - \gamma^a_c E^c{}_{;d} u^d u_b - u^a u_b \dot{u}_c E^c$$

and then notice that $E^c{}_{;d} u^d = E^c{}_{\wedge d} u^d + E^e \hat{K}_e{}^c{}_d u^d = \mathcal{D}E^c + E^e(\dot{u}_e u^c + u^c{}_{;e}) = P^c_\gamma \dot{E}^\gamma + (\dot{U}_\alpha E^\alpha) u^c + u^c{}_{:e} E^e$. If we substitute in the above equation using $D_{\alpha\beta} = g_{\alpha\beta}(\tau)$ as the components of the deformation, $u_{c:e} = \theta_{ce} + \omega_{ce}$, $\theta_{\gamma\epsilon} = \theta_{ce} P^c_\gamma P^e_\epsilon = \frac{1}{2} \dot{D}_{\gamma\epsilon}$, $\omega_{ce} = \varepsilon_{ced} \nu^d = \frac{1}{\rho} \hat{\eta}^*_{ced} \nu^d = \frac{1}{\rho} \eta^*_{\gamma\epsilon\delta} \nu^\delta P^\gamma_c P^\epsilon_e$, and using

$D_{\alpha\beta}$, a function of τ, to raise and lower indices on the body manifold we have

$$E^a{}_{;b} = P^a_\alpha P^\beta_b E^\alpha{}_{;\beta} + \frac{1}{2} u^a E^\gamma \dot{D}_{\gamma\beta} P^\beta_b + \frac{1}{\rho} u^a E^\gamma \overset{*}{\eta}_{\gamma\beta\delta} \nu^\delta P^\beta_b$$

$$- P^a_\gamma \dot{E}^\gamma u_b + \frac{1}{\rho} P^a_\alpha D^{\alpha\delta} u_b E^\epsilon \overset{*}{\eta}_{\delta\epsilon\gamma} \nu^\gamma + \frac{1}{2} P^a_\alpha D^{\alpha\delta} u_b E^\epsilon \dot{D}_{\delta\epsilon} - u^a u_b \dot{U}_\gamma E^\gamma .$$

Thus we have the covariant derivative of the orthogonal vector field E^a in terms of quantities on the body manifold we can identify using the general theory of a thermodynamic material element with slightly extended state and configuration spaces. The same equation with index a lowered is

$$E_{a;b} = P^\alpha_a P^\beta_b E_{\alpha;\beta} + \frac{1}{2} u_a E^\gamma \dot{D}_{\gamma\beta} P^\beta_b + \frac{1}{\rho} u_a E^\gamma \overset{*}{\eta}_{\gamma\beta\delta} \nu^\delta P^\beta_b$$

$$- P^\gamma_a D_{\gamma\alpha} \dot{E}^\alpha u_b + \frac{1}{\rho} P^\alpha_a u_b E^\epsilon \overset{*}{\eta}_{\alpha\epsilon\gamma} \nu^\gamma + \frac{1}{2} P^\alpha_a u_b E^\epsilon \dot{D}_{\alpha\epsilon} - u_a u_b \dot{U}^\gamma E_\gamma .$$

There are similar equations for $H_{a;b}$, $B^a{}_{;b}$, $D^a{}_{;b}$ etc. It is worth mentioning here that $\dot{U}_\gamma = \dot{u}_c P^c_\gamma$, that Greek indices are raised and lowered on the body using $D_{\alpha\beta} = g_{\alpha\beta}(\tau)$, and $D_{\alpha\beta}$ should not be confused with the deformation rate D in Chapter I. Substituting back we obtain Maxwell's equations on the body manifold as

$$-\rho_{(c)} = 2\nu^\gamma H_\gamma - D^\alpha{}_{;\alpha} ,$$

$$j^\epsilon = \rho \hat{\eta}^{\epsilon\gamma\delta}(H_{\gamma;\delta} + H_\gamma \dot{U}_\delta) - \dot{D}^\epsilon + \frac{2}{\rho} D^{\epsilon\alpha} \overset{*}{\eta}_{\alpha\beta\gamma} D^\beta \nu^\gamma + D^{\epsilon\alpha} \dot{D}_{\alpha\beta} D^\beta - D^\epsilon \theta ,$$

$$0 = 2\nu^\alpha E_\alpha + B^\epsilon{}_{;\epsilon} ,$$

$$0 = \rho \hat{\eta}^{\beta\alpha\gamma}(E_{\alpha;\gamma} + E_\alpha \dot{U}_\gamma) + \dot{B}^\beta - \frac{2}{\rho} D^{\beta\alpha} \overset{*}{\eta}_{\alpha\delta\gamma} B^\delta \nu^\gamma - D^{\beta\alpha} \dot{D}_{\alpha\gamma} B^\gamma + B^\beta \theta .$$

Here $E^\epsilon = \dfrac{dE^\epsilon}{d\tau}$ as these body manifold quantities are functions of proper time τ.

We could extend the configuration space G so that an element $G \in G$

would be of the form $G = (D,H,\Theta,g,\dot{U},\vec{\upsilon},\vec{J},\rho_{(c)})$ where $\vec{\upsilon} \in \mathcal{T}$ is the vorticity, $\vec{J} \in \mathcal{T}$ is the current density, and $\rho_{(c)} \in \mathbb{R}$ is the charge density. Likewise an element of the stress space $R \in \mathcal{S}$ would be $R = (S,\eta,\psi,\vec{E},\vec{H},\chi,\mu)$.

CHAPTER V

RELATION OF CONSTITUTIVE AND FIELD EQUATIONS

(V.1) The Unidirectional Body Manifold.

In Chapter III we formed the field equations completely for the Unidirectional space-time in adapted frame components in a local neighborhood. Here we would like to introduce a frame component system on the body manifold B (of three dimensions) defined locally, which is consistent with the group structure on B (i.e. is obtained from a reference chart of $\mathcal{U}_B$ and a fixed basis of V_B, c.f. (IV.6)). We suppose that $U \subset B$ is this open neighborhood, $X \in U$, the three vector fields on U are written as X^α, Y^α, Z^α where $\alpha = \mathit{1}, \mathit{2}, \mathit{3}$ are the italicized numbers referring to the Body manifold, the variable being a lower case Greek index. As before, for the space time, the tetrad is u^a, r^a, s^a, t^a where $a = 1, 2, 3, 4$ are the regular numbers, the variable being a Latin index. Of course in frame components $u^a = \delta^{a4}$, $r^a = \delta^{a1}$, $s^a = \delta^{a2}$, $t^a = \delta^{a3}$ and in the case of the body $X^\alpha = \delta^{\alpha \mathit{1}}$, $Y^\alpha = \delta^{\alpha \mathit{2}}$, $Z^\alpha = \delta^{\alpha \mathit{3}}$. The Ricci coefficients for the space-time are $T_j{}^i{}_k$ or $T_b{}^a{}_c$ and for the body $T_\beta{}^\alpha{}_\gamma$. There is no confusion here about specific components, $T_2{}^1{}_3$ is for space-time, $T_{\mathit{2}}{}^{\mathit{1}}{}_{\mathit{3}}$ for the body.

If X^α, Y^α, Z^α are vector fields defined on $U \subset B$ determining a frame component system, then there is a naturally defined material connection on U which has zero Riemann tensor. It is material because its parallel transports are material isomorphisms, since we have constructed this frame component system out of a material reference

chart on the group structure on B. We define this connection by the conditions on its covariant derivative $|$ namely,

$$X^{\alpha}{}_{|\beta} = 0 = Y^{\alpha}{}_{|\beta} = Z^{\alpha}{}_{|\beta}. \qquad \text{(V.1.1)}$$

Thus it parallel transports the frame defining the frame component system to itself along any curve in U. Thus its holonomy groups are trivial and so we expect the curvature tensor to be zero. This can be seen another way. By (V.1.1), $0 = X^{\alpha}{}_{|\beta} = X^{\alpha}{}_{,\beta} + X^{\gamma}\Gamma_{\gamma}{}^{\alpha}{}_{\beta}$ so $\Gamma_{1}{}^{\alpha}{}_{\beta} = 0$ since $X^{\alpha}{}_{,\beta} = 0$ in frame components, and similarly $\Gamma_{2}{}^{\alpha}{}_{\beta} = \Gamma_{3}{}^{\alpha}{}_{\beta} = 0$, so $\Gamma_{\gamma}{}^{\alpha}{}_{\beta} = 0$ in this frame system identically. Therefore we see that $R^{\alpha}{}_{\beta\gamma\delta} = \Gamma_{\beta}{}^{\alpha}{}_{\delta,\gamma} - \Gamma_{\beta}{}^{\alpha}{}_{\gamma,\delta} + \Gamma_{\sigma}{}^{\alpha}{}_{\gamma}\Gamma_{\beta}{}^{\sigma}{}_{\delta} - \Gamma_{\sigma}{}^{\alpha}{}_{\delta}\Gamma_{\beta}{}^{\sigma}{}_{\gamma} + \Gamma_{\beta}{}^{\alpha}{}_{\sigma}T_{\gamma}{}^{\sigma}{}_{\delta}$ implies $R^{\alpha}{}_{\beta\gamma\delta} = 0$. The components of the torsion tensor for this connection in frame components are $-T_{\beta}{}^{\alpha}{}_{\gamma}$ (c.f. (II.20)).

The chart with domain U which determines this system is said to be *unidirectional* if the Ricci coefficients $T_{\beta}{}^{\alpha}{}_{\gamma}$ single out just one spatial direction which we take to be X^{α}. The naturally defined mass element[1] $\overset{*}{\eta}_{\alpha\beta\gamma}$ (see (I.11)) has components in these frame components given by $\overset{*}{\eta}_{123} = -1$, so $\overset{*}{\eta}_{\alpha\beta\gamma} = -\epsilon_{\alpha\beta\gamma}$. We have a Riemannian metric tensor defined on U in a natural way so as to make the frame components orthonormal. This metric is a property of the frame component system and not the body manifold B. It is preserved by the special material connection we have described here on $U \subset B$ but not necessarily by other material connections. It is useful for switching to the covector basis, and as a reference for the deformation measurements. We denote it by $g_{\alpha\beta} = \delta_{\alpha\beta}$ in these components. It differs from the proper time

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[1] Recall that the symmetry group of a material element must be a subgroup of the special linear group.

dependent metric $g_{\alpha\beta}(\tau)$ at $X \in B$ defined as in (I.9) projected down from space-time. The mass element ($\eta^{*}_{\alpha\beta\gamma}$ in components) is given by the exterior product $-3!\ \partial^{1} \wedge \partial^{2} \wedge \partial^{3}$ of the three imperfect differentials (III.6) which form the dual basis in this frame component system, i.e. $\partial^{1}_{\alpha} = X_{\alpha}$, $\partial^{2}_{\alpha} = Y_{\alpha}$, $\partial^{3}_{\alpha} = Z_{\alpha}$ where $g_{\alpha\beta} = \delta_{\alpha\beta}$ is used to lower indices. Hence $\partial^{\alpha}_{\beta} = \delta^{\alpha}_{\beta}$ is the Kronecker delta.

We put $A_{\alpha\beta} = \epsilon_{\alpha\beta\gamma}X^{\gamma} = Y_{\alpha}Z_{\beta} - Z_{\alpha}Y_{\beta}$. Then the unidirectional Ricci coefficients can be written out as $T_{\beta}{}^{\alpha}{}_{\gamma} = wX^{\alpha}A_{\beta\gamma} + v\epsilon_{\beta}{}^{\alpha}{}_{\gamma}$ for three dimensions, where $w_{,\alpha} = \overset{*}{w}X_{\alpha}$ and $v_{,\alpha} = \overset{*}{v}X_{\alpha}$, i.e. $w_{,2} = w_{,3} = v_{,2} = v_{,3} = 0$, $w_{,1} = \overset{*}{w}$, $v_{,1} = \overset{*}{v}$. The integrability conditions $w_{,\alpha\beta} - w_{,\beta\alpha} = T_{\alpha}{}^{\gamma}{}_{\beta}w_{,\gamma}$ and $v_{,\alpha\beta} - v_{,\beta\alpha} = T_{\alpha}{}^{\gamma}{}_{\beta}v_{,\gamma}$ imply either v and w are both constants or else $T_{2}{}^{1}{}_{3} = 0$. This latter condition, $T_{2}{}^{1}{}_{3} = 0$ is equivalent to $w = v$. The Jacobi identity implies $T_{2}{}^{1}{}_{3,1} = 0$ and is itself therefore implied, in this case, by the integrability conditions. Hence either $w = v$ and there is no restriction on the derivatives or else $w \neq v$ and they are both constant. If we have $w = 0$ and v a constant with $T_{\beta}{}^{\alpha}{}_{\gamma} = v\epsilon_{\beta}{}^{\alpha}{}_{\gamma}$ the chart with domain U is said to be *omnidirectional*. If B is covered by a family of omni(uni) directional bundle or reference charts from the group structure then we say the body manifold B is *omni(uni) directional* respectively.

(V.2) Motion of a Unidirectional Body in Unidirectional Space-Time.

In this section we will consider evaluating the components of the projection tensor P^{α}_{a} and its inverse P^{a}_{α} in mixed frame components. Using the results in (I.27) we have the material derivative of the mixed tensor extended to frame components as,

$$0 = P^{\alpha}_{b\wedge c} = P^{\alpha}_{b,c} + P^{\beta}_{b}\Gamma_{\beta}{}^{\alpha}{}_{\gamma}P^{\gamma}_{c} - P^{\alpha}_{a}\hat{\Gamma}_{b}{}^{a}{}_{c}\ ,$$

so $P^\alpha_{b,c} = P^\alpha_a \hat{\Gamma}_b{}^a{}_c$ since the components $\Gamma_\beta{}^\alpha{}_\gamma$ of the material connection are zero in frame components. Using the equation in (I.27) for the torsion of the material connection in terms of the lift of the torsion on the body, and the result in (II.20) that $\hat{\Gamma}_b{}^a{}_c - \hat{\Gamma}_c{}^a{}_b$ is the sum of the space-time coordinate torsion and the torsion tensor in these generalized coordinates (i.e. frame components) we have that

$$\hat{\Gamma}_b{}^a{}_c - \hat{\Gamma}_c{}^a{}_b = T_b{}^a{}_c + P^a_\alpha P^\beta_b P^\gamma_c(-T_\beta{}^\alpha{}_\gamma) + u^a(u_{c;b}-u_{b;c})$$

since $-T_\beta{}^\alpha{}_\gamma$ are the components of the torsion tensor of the material connection on the body. Hence we see that

$$P^\alpha_{b,c} - P^\alpha_{c,b} = P^\alpha_a T_b{}^a{}_c - P^\beta_b P^\gamma_c T_\beta{}^\alpha{}_\gamma \qquad \text{(V.2.1)}$$

is the integrability condition for P^α_b in frame components.

If we are to consider a unidirectional body manifold in a unidirectional space-time, there must be a correspondence of the singled out directions, and a lack of specification of the other two. More precisely, the vectors X^α, Y^α, Z^α must be orthogonal in $g_{\alpha\beta}(\tau)$ with Y^α and Z^α of equal length in this metric for each τ. Thus we write out the frame components of the mixed tensors as follows.

Components of P^α_a: $P^\alpha_4 = 0$, $\alpha = 1,2,3$, $P^1_1 = a > 0$, $P^2_2 = P^3_3 = b \cos\theta$, $P^2_3 = -P^3_2 = b \sin\theta$, $b > 0$.

Components of P^a_α: $P^4_\alpha = 0$, $\alpha = 1,2,3$, $P^1_1 = 1/a$, $P^2_2 = P^3_3 = \frac{1}{b}\cos\theta$, $P^2_3 = -P^3_2 = -\frac{1}{b}\sin\theta$.

The other components not listed are zero. Thus the relations $P^\alpha_a P^a_\beta = \delta^\alpha_\beta$ and $P^\alpha_a P^b_\alpha = \gamma^b_a = \delta^b_a + u^b u_a$ where $u^4 = 1$, $u^1 = u^2 = u^3 = 0$, $u_4 = -1$, will hold. Of course a, b and θ are functions[1] which

[1] The function θ is not to be confused with the expansion rate θ.

have 1 and 4 derivatives only, i.e. $a_{,2} = a_{,3} = b_{,2} = \cdots = 0$.

The next step is to take the Ricci coefficients $T_b{}^a{}_c$ in space-time given by the 10 function form in (III.15) and the unidirectional Ricci coefficients for the body manifold given by $T_\beta{}^\alpha{}_\gamma = wX^\alpha A_{\beta\gamma} + v\epsilon_\beta{}^\alpha{}_\gamma$ and substitute into the integrability condition (V.2.1) using the forms just given for P^α_a. We find that (if τ is defined on world lines)

$$\frac{da}{d\tau} = a_{,4} = -\dot{a} = -\frac{a}{2}(f+t), \quad a(h-\hat{f}) = b^2(w-v),$$

$$b\theta' \sin\theta - b' \cos\theta = -\frac{r}{2} b \cos\theta + \left(\frac{q}{2} - \hat{f}\right) b \sin\theta + avb \sin\theta,$$

$$b\theta' \cos\theta + b' \sin\theta = \left(\frac{q}{2} - \hat{f}\right) b \cos\theta + \frac{r}{2} b \sin\theta + avb \cos\theta,$$

$$b\dot{\theta} \sin\theta - \dot{b} \cos\theta = \left(\frac{k}{2} - p\right) b \sin\theta - \frac{f}{2} b \cos\theta,$$

$$b\dot{\theta} \cos\theta + \dot{b} \sin\theta = \left(\frac{k}{2} - p\right) b \cos\theta + \frac{f}{2} b \sin\theta.$$

These are the complete equations we obtain by direct substitution. The trigonometric forms can be simplified by recombining the equations above and we end up with the conditions:

$$\begin{aligned} \dot{a} &= \frac{a}{2}(f+t), & a(h-\hat{f}) &= b^2(w-v), \\ \theta' &= \frac{q}{2} - \hat{f} + av, & b' &= \frac{r}{2} b, \\ \dot{\theta} &= \frac{k}{2} - p, & \dot{b} &= \frac{f}{2} b. \end{aligned} \tag{V.2.2}$$

We should remark again, that $\dot{\theta} = -\frac{d\theta}{d\tau} = -\theta_{,4}$ according to our convention, and similarly for $\dot{b}$. Then (V.2.2) is the statement of the integrability condition (V.2.1) for the unidirectional case. It is the basic and fundamental condition that tells us when a unidirectional space-time is describing the motion of a body manifold. We should observe that $\dot{w} = \dot{v} = 0$ and $w' = w_{,1} = P^1_1 w_{,1} = aw_{,1}$ and $v' = av_{,1}$, but that unless $w = v$, it must be that $w_{,1} = v_{,1} = 0$. Of course always we have $a > 0$

and $b > 0$, so in particular these are non-zero. From (V.2.2) we see that $w = v$ if and only if $h = \hat{f}$, so we require $h = \hat{f}$ if w and v are to have non-zero derivatives.

(V.3) The Integrability Conditions.

In this section we wish to examine the conditions imposed on the Unidirectional space-time and its 10 functions in (III.15) by the requirement that it represents the motion of a unidirectional body manifold. This means we must impose the integrability conditions (III.15.1) on the functions a, b, and θ to show that P^{α}_{a} exists. If we write $\bar{b} = \ell n\, b$ so $\dot{\bar{b}} = \frac{f}{2}$ and $\bar{b}' = \frac{r}{2}$ and impose (III.15.1) we see that in light of (III.16.1) the condition is $f(p-g) + r(h-\hat{f}) = 0$. If we apply the integrability condition for θ we find that substituting for a from (V.2.2) into it gives us the Jacobi identity (III.16.3). Hence we need only impose $0 = \left(\frac{q}{2} - \hat{f} + av\right)(h-\hat{f}) + \left(\frac{k}{2} - p\right)(p-g)$. For the case of a we have no condition on a' in (V.2.2), however we can take the equation $a(h-\hat{f}) = b^{2}(w-v)$ and take dot derivatives and substitute and obtain $\dot{h} - \dot{\hat{f}} = -\frac{1}{2}(t-f)(h-\hat{f})$ which is (III.16.7). Thus we see there is no problem with the integrability for a and we end up with two conditions

$$\begin{aligned} &f(p-g) + r(h-\hat{f}) = 0 \\ &\left(\frac{k}{2} - p\right)(p-g) + \left(\frac{q}{2} - \hat{f} + av\right)(h-\hat{f}) = 0 \end{aligned} \qquad \text{(V.3.1)}$$

(V.4) The Volume Element and Material Contorsion.

If we write the equation $\varepsilon_{ijk} = \frac{1}{\rho}\overset{*}{\eta}_{ijk}$ from (I.11) in frame components we find $-\epsilon_{abc} = \varepsilon_{abc} = \frac{1}{\rho}\overset{*}{\eta}_{abc} = \frac{1}{\rho}\overset{*}{\eta}_{\alpha\beta\gamma}P^{\alpha}_{a}P^{\beta}_{b}P^{\gamma}_{c} = -\frac{1}{\rho}\epsilon_{\alpha\beta\gamma}P^{\alpha}_{a}P^{\beta}_{b}P^{\gamma}_{c}$. In the unidirectional case this implies $\rho = ab^{2}$. Clearly, using (V.2.2) we see that $\dot{\rho} = \rho\theta$, so recalling $\dot{\rho} = -\frac{d\rho}{d\tau}$ we

have $(\rho u^a)_{;a} = 0$ as expected. In determining this we used the standard result $\theta = \dfrac{3f + t}{2}$ for a unidirectional space-time (III.15).

Now we have $0 = u_{a\wedge c} = u_{a,c} - u_d \hat{\Gamma}^d{}_{a\,c}$ so in frame components, using the result $P^\alpha_{b,c} = P^\alpha_a \hat{\Gamma}^a{}_{b\,c}$ from (V.2) we find that $\hat{\Gamma}^a{}_{b\,c} = P^a_\alpha P^\alpha_{b,c}$ using the mixed frame component system notation. From the unidirectional mixed tensor components in (V.2) we have $\hat{\Gamma}^1{}_{1\,1} = \dfrac{a'}{a}$, $\hat{\Gamma}^1{}_{1\,4} = -\dfrac{\dot{a}}{a} = -\dfrac{(f+t)}{2}$, $\hat{\Gamma}^2{}_{2\,1} = \hat{\Gamma}^3{}_{3\,1} = \dfrac{b'}{b} = \dfrac{r}{2}$, $\hat{\Gamma}^2{}_{2\,4} = \hat{\Gamma}^3{}_{3\,4} = -\dfrac{\dot{b}}{b} = -\dfrac{f}{2}$, $\hat{\Gamma}^2{}_{3\,1} = -\hat{\Gamma}^3{}_{2\,1} = \theta' = \dfrac{q}{2} - \hat{f} + av$, $\hat{\Gamma}^3{}_{2\,4} = -\hat{\Gamma}^2{}_{3\,4} = \dot{\theta} = \dfrac{k}{2} - p$, and all others zero. We can therefore write (with $\Lambda^a{}_b$ as in III.15),

$$\hat{\Gamma}^a{}_{b\,c} = \left(\frac{a'}{a} - \frac{r}{2}\right) r^a r_b r_c + \frac{t}{2}\, r^a r_b u_c + \frac{r}{2}\,\gamma^a_b r_c + \frac{f}{2}\,\gamma^a_b u_c$$
$$+ \left(\frac{q}{2} - \hat{f} + av\right)\Lambda^a{}_b r_c + \left(\frac{k}{2} - p\right)\Lambda^a{}_b u_c\ .$$

From this the material contorsion is given by $\hat{K}_b{}^a{}_c = \left\{{a \atop b\ c}\right\} - \hat{\Gamma}^a{}_{b\,c}$ where $\left\{{a \atop b\ c}\right\} = \frac{1}{2}(T_b{}^a{}_c - T^a{}_{bc} - T^a{}_{cb})$ and $T_b{}^a{}_c$ is the same as in (III.15). Of course this is a special type of material connection, namely a lift of a locally defined material connection on B which has zero curvature. Nevertheless, the unidirectional symmetry of the body is made manifest through this connection.

It is easy to check that $\hat{K}_b{}^a{}_4 = u^a{}_{;b} + u^{a\cdot} u_b$, $\hat{K}_b{}^4{}_c = u_{b;c}$ and $\hat{K}_4{}^a{}_c = u^a{}_{;c}$ which is in agreement with the expressions in (I.22). Thus we can orthogonalize the material contorsion to obtain the delta tensor as

$$\hat{\Delta}_{bac} = \left(\frac{r}{2} - \frac{a'}{a}\right) r_a r_b r_c + \frac{\hat{f}}{2}\,\epsilon_{bac} - \frac{r}{2}\,\gamma_{ac} r_b + \frac{r}{2}\,\gamma_{bc} r_a$$
$$- \frac{r}{2}\,\gamma_{ab} r_c + \frac{h}{2}\, r_a \Lambda_{bc} - \frac{h}{2}\, r_b \Lambda_{ac} - \left(\frac{h}{2} - \hat{f} + av\right)\Lambda_{ab} r_c\ .$$

Now $g_{ab\wedge c} = \eta_{ab\wedge c} = \gamma_{ab\wedge c} = \gamma_{ab,c} - \gamma_{db}\hat{\Gamma}^d{}_{a\,c} - \gamma_{da}\hat{\Gamma}^d{}_{b\,c}$
$= 2\check{\theta}_{abc}$ from (I.24) using frame components.

Thus $\breve{\theta}_{abc} = -\hat{\Gamma}_{(ba)c}$ and so we find the generalized stretch to be

$$\breve{\theta}_{abc} = \left(\frac{r}{2} - \frac{a'}{a}\right) r_a r_b r_c - \frac{t}{2} r_a r_b u_c - \frac{r}{2} \gamma_{ab} r_c - \frac{f}{2} \gamma_{ab} u_c .$$

Of course $\breve{\theta}_{abc} u^c = \frac{f}{2} \gamma_{ab} + \frac{t}{2} r_a r_b = \theta_{ab}$ and the orthogonalized stretch $\theta_{abc} = \breve{\theta}_{abd} \gamma^d_c$ is $\theta_{abc} = \left(\frac{r}{2} - \frac{a'}{a}\right) r_a r_b r_c - \frac{r}{2} \gamma_{ab} r_c$, and of course $\theta_{abc} = \hat{\Delta}_{(ba)c}$ since $\breve{\theta}_{abc} = \hat{K}_{(ba)c}$.

Consider the results of (I.26) for a moment. We see that $\breve{\theta}_e = \breve{\theta}^a_{\ a e} = \left(\frac{r}{2} - \frac{a'}{a}\right) r_e - \frac{t}{2} u_e - \frac{3r}{2} r_e - \frac{3f}{2} u_e$, and so we can check immediately that $\breve{\theta}_e = -(\ell n\, \rho)_{,e}$ where $\rho = ab^2$. Thus the condition for the density to be constant along directions orthogonal to u^a is $\rho' = 0$, or $a' = -ra$. If the density is to be constant along the past null cone (observed density) then $\rho' = 0 = \dot{\rho}$ is required. If $a' + ra = 0$ were imposed, then the conditions (III.15.1) would imply for this special case that, among other things, $f' + t' + 2\dot{r} = (f+t)(r+s/2)$.

As well, the generalized rotation $\omega_{abc} = -\Delta_{abc} - \omega_{ab} u_c$ can be determined (see (I.24)). Of course $\Delta_{abc} = \hat{\Delta}_{[ab]c}$ is the orthogonalized fundamental contorsion and is given by

$$\Delta_{bac} = \frac{\hat{f}}{2} \epsilon_{bac} - \frac{r}{2} \gamma_{ac} r_b + \frac{r}{2} \gamma_{bc} r_a + \frac{h}{2} r_a \Lambda_{bc} - \frac{h}{2} r_b \Lambda_{ac}$$
$$- \left(\frac{h}{2} - \hat{f} + av\right) \Lambda_{ab} r_c .$$

(V.5) <u>The Omnidirectional Case</u>.

We discuss here the motion of an omnidirectional body in an omnidirectional space-time. Then in the expression for $T_j{}^i{}_k$ in (III.15) all functions are zero except for f and $\hat{f}$ and these depend only on time, i.e. $f' = \hat{f}' = 0$. For an omnidirectional body, $w = 0$, v = constant and for an omnidirectional motion $\theta = 0$, $b = a$ and $a' = b' = 0$. Then from (V.2.2), $\dot{a} = \frac{a}{2} f$ and $\hat{f} = av$ so the Jacobi identity

$\dot{\hat{f}} = \frac{1}{2} f\hat{f}$ gives us the triviality $f(av - \hat{f}) = 0$. Therefore the two conditions $\hat{f} = av$, v = constant, and $\dot{a} = \frac{a}{2} f$ completely solve the problem for the omnidirectional case. Now the density is $\rho = ab^2 = a^3$ so $\rho c^2 = a^3 c^2$ and using $\rho c^2 + \epsilon = \frac{3}{4\kappa}(\hat{f}^2 + f^2)$ from (III.12) we have that $\epsilon = \frac{3}{4\kappa}(\hat{f}^2 + f^2) - a^3 c^2$ gives us the internal energy density. Recall that always $a > 0$. For a dust solution where $P = \frac{1}{4\kappa}(4\dot{f} - 3f^2 - \hat{f}^2) = 0$ we have $0 = (\epsilon u^a)_{;a} = \epsilon\theta - \dot{\epsilon}$ so that $\dot{\epsilon} = \frac{3f}{2}\epsilon$ which can be satisfied by taking $\epsilon = 0$ which is true for an appropriate choice of v, if $\hat{f} \neq 0$. With nonzero pressure, $\dot{\epsilon} = \frac{3}{2} f(P + \epsilon) = -\frac{d\epsilon}{d\tau}$, so that $f < 0$ means increasing internal energy, and $f > 0$ the opposite.

Let us look now at the physically omnidirectional geometrically unidirectional solutions of (III.19) and (III.20). If we take the general unidirectional solution (III.15) and impose only the conditions $p = g$, $\hat{f} = h$, $t = s = 0$, and the perfect fluid condition $r' = 0$ and zero heat flow condition $f' = 0$ (see (III.18)) we obtain the pressure and density of (III.20) with the two Jacobi identities $\dot{r} = \frac{1}{2} rf$ and $k' - \dot{q} + q\frac{f}{2} + 2\dot{h} - hf - 2p' = 0$. The basic integrability condition $\phi^{\cdot\,\prime} - \phi^{\prime\,\cdot} = -\frac{1}{2}\phi' f$ must be satisfies if ϕ is replaced by any of the functions f, r, k, q, h etc. Viewing this as the motion of a unidirectional body manifold we see from (V.2.2) that $w = v$, $\dot{a} = a\frac{f}{2}$, and θ', $\dot{\theta}$, b', $\dot{b}$ are as given in (V.2.2) and are integrable to θ and b. Thus we have a fair degree of generality for θ which is in essence the rotational position of the space frame at time τ relative to a fixed reference placement of $X \in B$ determined by the frame field spanning B_X, the basis for our frame component system on $U \subset B$. This is true even though the material is irrotational i.e. $\omega_{ab} = 0$. To understand this difference we need to realize that ω_{ab} measures rotational deviation from the Fermi-transported frame along $P^{-1}(X)$,

and the frame r^a, s^a, t^a is not necessarily Fermi transported. For $s = t = 0$, $g = p$, the condition that r^a, s^a, t^a be flow Fermi transported along world lines is $\frac{k}{2} = p$ or $\dot{\theta} = 0$.

We have seen that $P = \frac{1}{4\kappa}(4\dot{f} - 3f^2 + r^2)$ and $\rho c^2 + \epsilon = \frac{3}{4\kappa}(f^2 - r^2)$ have zero derivatives along r^a. If $\rho = ab^2$ is also to satisfy $\rho' = 0$ then from (V.4), $a' + ra = 0$ and the integrability condition $2\dot{r} = fr$ (for $s = t = 0$)) is already satisfied, so a is integrable. In particular then, the mixed projection tensor is not omnidirectional, but a stretching occurs along r^a and $\theta_{abc}r^c = \frac{3r}{2} r_a r_b - \frac{r}{2}\gamma_{ab}$. Of course, since $\rho' = 0$, this stretching is isochoric, i.e. $\gamma^{ab}\theta_{abc} = 0$.

(V.6) Solution for Extended Mass Sheet of Constant Thickness.

For this mass slab we shall impose the condition of zero rotation or vorticity, i.e. $g = p$ on the unidirectional weakly flow-static solutions of (III.22). Thus instead of the dust solutions which are supported by rotation but have zero acceleration, we have acceleration and of course pressure, and gravity is felt in the local rest frames. Putting $g = p$ we obtain

$$\sigma_2 = \sigma_3 = -\frac{1}{2\kappa}\left(-r' - s' + \frac{r^2}{r} + \frac{s^2}{2} + \frac{rs}{2}\right),$$

$$\sigma_1 = -\frac{1}{2\kappa}\left(\frac{r^2}{2} + rs\right), \quad \rho c^2 + \epsilon = \frac{-1}{2\kappa}\left(-2r' + \frac{3r^2}{2}\right), \quad \lambda = 0.$$

Putting $P = -\sigma_1 = -\sigma_2 = -\sigma_3$ we have $P = \frac{1}{2\kappa}\left(\frac{r^2}{2} + rs\right)$, $r' + s' = \frac{s}{2}(s-r)$ and $\rho c^2 + \epsilon = \frac{1}{2\kappa}\left(2r' - \frac{3r^2}{2}\right)$.

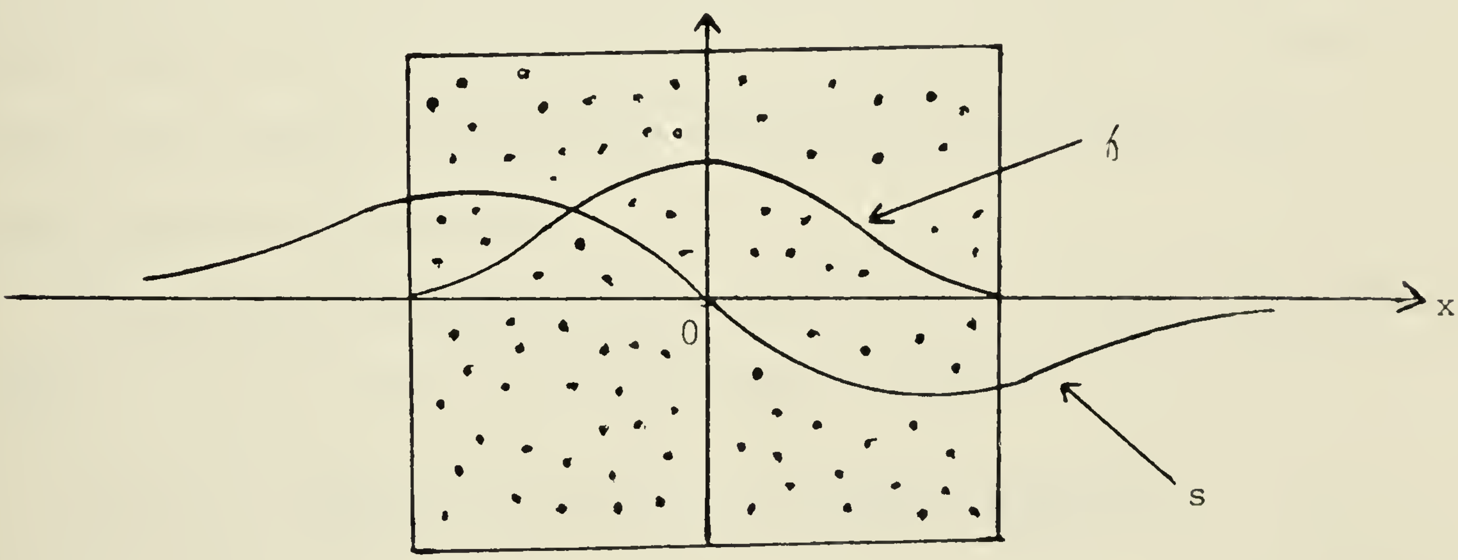
As an example, we try the equation of state $P = c_0(\rho c^2 + \epsilon)$ where c_0 is a function satisfying $0 < c_0 < 1$. Although we have oversimplified and ignored thermodynamic effects, this equation covers a surprizingly wide range of interesting physical applications as we shall see. This

gives the added equation $r' = \frac{1}{4}\left(3 + \frac{1}{c_0}\right) r^2 + \frac{rs}{2c_0}$ in addition to $r' + s' = \frac{s}{2}(s-r)$. The Jacobi identities in (III.22) automatically hold if we put $k = 2p$. It is worthy of note that there are vacuum solutions if $r = 0$, $s' = s^2/2$ or $s = -\frac{r}{2}$ and $r' = \frac{3}{4} r^2$.

If we put $\mathfrak{f} = \frac{r^2}{2} + rs$ so that $P = \frac{1}{2\kappa}\mathfrak{f}$ and $\rho c^2 + \epsilon = \frac{\mathfrak{f}}{2\kappa c_0}$ then we obtain the two equations[1] (in $x > 0$ range to avoid frame component breakdown)

$$\mathfrak{f}' = \frac{1}{2}\left(1 + \frac{1}{c_0}\right) s\mathfrak{f}, \quad s' + \frac{1}{2}\left(3 + \frac{1}{c_0}\right)\mathfrak{f} = \pm s\sqrt{s^2 + 2\mathfrak{f}} - \frac{s^2}{2} \tag{V.6.1}$$

where $\mathfrak{f}$ represents the mass-energy density and pressure, and s represents the acceleration ($\dot{u}^a = -\frac{s}{2} r^a$). If only r and s are nonzero in the expression for $T_j{}^i{}_k$ in (III.15), then $T_j{}^1{}_k = 0$ so $dr_{a|b} = 0$ meaning the length coordinate x exists for our slab locally and not just along integral curves of r^a.



Considering $x = 0$ as the centre of the slab, the $\mathfrak{f}'$ equation tells us that $s < 0$ for $x > 0$ and conversely, $s > 0$ for $x < 0$ since $\mathfrak{f}$ achieves a maximum at $x = 0$ on physical grounds. Since $\dot{u}^a = -\frac{s}{2} r^a$ this means that for $x > 0$ the acceleration is directed away from the centre of the slab in the rest frame. This is reasonable,

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[1] The odd function r has finite discontinuity at $x = 0$.

since on the earth's surface we seem to be accelerating away from the centre of the earth at a rate of 9.8 meters per second squared. Also for $x < 0$, the acceleration is directed in the $-r^a$ direction, again away from the centre. By symmetry $\mathfrak{f}(x) = \mathfrak{f}(-x)$ and $s(x) = -s(-x)$. This means that for the two possible exterior vacuum solutions (one with $s' = \frac{s^2}{2}$, another with $s' = -\frac{3}{2}s^2$) we must choose $s' = \frac{s^2}{2}$ with solution $s = \frac{-2}{x + c_1}$, so that s has the correct sign for large positive or negative values of x. By symmetry the constant c_1 must be zero. This means also that in the s' equation in (V.6.1) we are required to take the minus sign $(x > 0)$. Furthermore we find the exterior vacuum solution with only s nonzero, and $s' = s^2/2$ to have the acceleration dependence $s \propto \frac{1}{x}$ for an extended slab as contrasted to the Newtonian constant acceleration. If we were going to match the exterior Schwarzschild solution for a spherical star to the interior case we would need to use bidirectional Ricci coefficients (III.26.1) since a spherically symmetric solution is physically unidirectional but geometrically bidirectional.

We can take the exterior vacuum solution for the spherical case which is static, namely the Schwarzschild solution

$$ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

and express it in a natural metric frame component system related to this coordinate system by rescaling the coordinates since the metric is diagonal. The transformation coefficients $v^i_{(a)} = G_{\gamma\alpha}{}^i{}_a$ are given by

$$v^1_{(1)} = \sqrt{1 - \frac{2m}{r}}, \quad v^2_{(2)} = \frac{1}{r}, \quad v^3_{(3)} = \frac{1}{r \sin\theta}, \quad v^4_{(4)} = \frac{1}{\sqrt{1 - \frac{2m}{r}}},$$

and all others are zero. The Ricci coefficients are given by the

spherically symmetric form (III.26.1) with

$$s = -g = \frac{2}{r}\sqrt{1-\frac{2m}{r}}\ , \quad h = -\frac{2\cos\theta}{r\sin\theta}\ , \quad k = \frac{2m}{r^2\sqrt{1-\frac{2m}{r}}}\ .$$

Since $dr_a\big|_b = 0$, i.e. $T_{a\ b}^{\ 1} = 0$, there is a local metric radial coordinate x with $\frac{dr}{dx} = \sqrt{1-\frac{2m}{r}}$, so that $x = m\cosh^{-1}\left(\frac{r}{m}-1\right) + \sqrt{r^2-2mr}$ plus a constant of integration which we can ignore. Furthermore $\theta_{,b} = \frac{1}{r}s_b$ in frame components, and $\phi_{,b} = \frac{1}{r\sin\theta}t_b$, the comma being frame component differentiation.

(V.7) Solution for Spherical Stars.

The techniques used for the extended mass slab in the last section, with boundary conditions matching to the exterior vacuum solution can be applied also to the physically interesting spherically symmetric case. It is easy to see that if the Schwarzschild values for s, g, h, k given in (V.6) are substituted in (III.26.1) then all the equations (III.26.2) through (III.26.11) are satisfied with zero mass density and pressure.

Using (III.26.6) and (III.26.7) in (III.26.5) we have $\rho c^2 + \epsilon = -\frac{1}{2\kappa}\left(k_1 - 3g_1 + \frac{3g^2}{2} + \frac{k^2}{2} - \frac{gk}{2} - sk\right)$, and of course $P = -\frac{1}{2\kappa}\left(g_1 - k_1 - \frac{g^2}{2} - \frac{k^2}{2} + \frac{gk}{2}\right)$. If we impose the equation of state $P = c_0(\rho c^2 + \epsilon)$, $c_0 = c_0(x)$, $0 < c_0 < 1$ then using (III.26.9), i.e. $(s = -g)$ we have,

$$(1+3c_0)g_1 - (1+c_0)k_1 = \frac{1}{2}(1+3c_0)g^2 + \frac{1}{2}(1+c_0)k^2 - \frac{1}{2}(1-c_0)gk, \qquad \text{(V.7.1)}$$

and g, k, s are functions only of the radial coordinate x, while h has also a nonzero 2-derivative.

Outside of the star we have functions θ and r defined with

$h = -\frac{2}{r}\cot\theta$ and $\theta_{,b} = \frac{1}{r}s_b$. We propose to extend them to the interior so as to maintain these equations, r of course being a function of x only. Integrability for θ implies $\frac{dr}{dx} = -\frac{gr}{2}$ and (III.26.10) and (III.26.11) hold. Of course $\frac{d(\ell n\ r)}{dx} = -\frac{g}{2}$ and $h_2 - \frac{h^2}{2} = \frac{2}{r^2}$ since $h_2 = \frac{2}{r^2}\csc^2\theta$. Substituting in (III.26.7) and combining with (V.7.1) we have (from $g_1 - k_1 = \frac{2}{r^2} + \frac{k^2}{2} + \frac{gk}{2}$) that

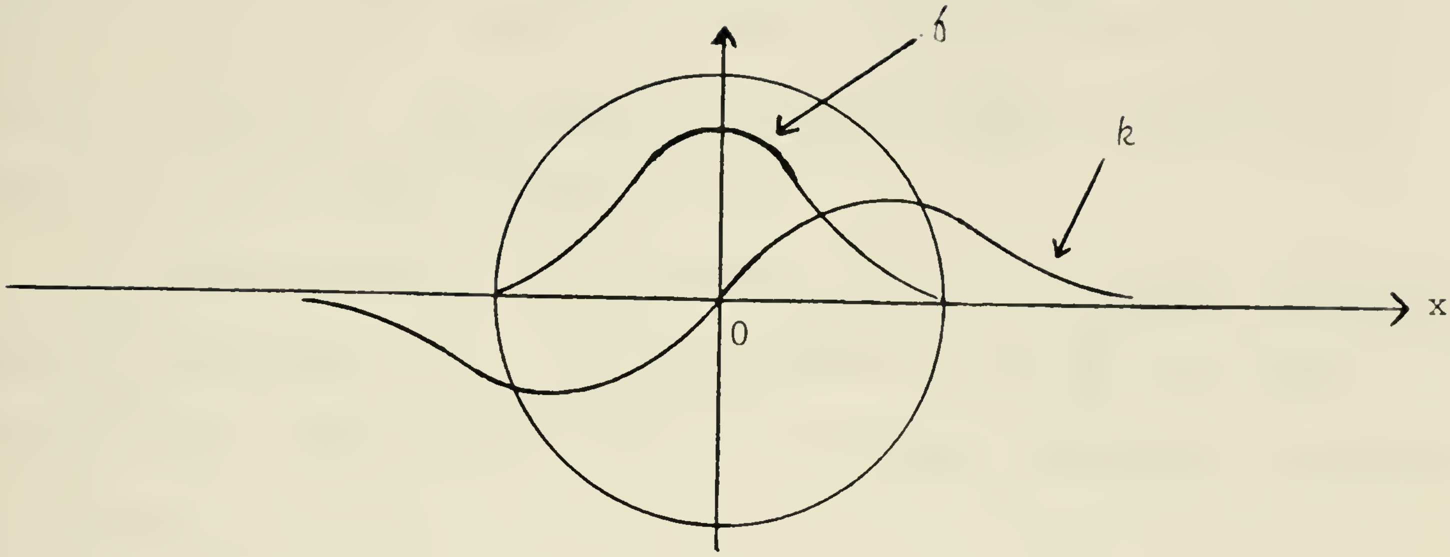
$$k_1 = \frac{(1+3c_0)}{4c_0}g^2 - \frac{1+3c_0}{c_0\ r^2} - \frac{k^2}{2} - \frac{(1+c_0)}{2c_0}gk$$
$$g_1 = \frac{(1+3c_0)}{4c_0}g^2 - \frac{1+c_0}{c_0\ r^2} - \frac{gk}{2c_0}, \quad r_1 = -\frac{gr}{2}, \tag{V.7.2}$$

where these three functions r, g, k depend only on the metric radial coordinate x and the subscript 1 indicates derivative with respect to x.

Putting $\delta = k_1 - g_1 + \frac{g^2}{2} + \frac{k^2}{2} - \frac{gk}{2}$ so $P = \frac{\delta}{2\kappa}$ and $\rho c^2 + \epsilon = \frac{\delta}{2\kappa c_0}$ we see that $\delta = \frac{g^2}{2} - \frac{2}{r^2} - gk$. Observe that $\frac{1+3c_0}{2c_0}\delta = k_1 + \frac{k^2}{2} - gk$ and also

$$\delta_1 = -\frac{(1+c_0)}{2c_0}k\delta. \tag{V.7.3}$$

This relation (3) is very important as it relates the change in mass density and pressure along the metric radial coordinate to the gravitational acceleration. Since $\dot{u}_a = \frac{k}{2}r_a$ here and $\dot{u}_a = -\frac{s}{2}r_a$ for the unidirectional case, the relation (3) above is exactly the same as its sister equation in (V.6.1). It is also worth noting that nowhere in this derivation have we explicitly needed to have c_0 as a constant, and these equations do not depend on derivatives of c_0.



We rewrite the differential equations in the form

$$f_1 = -\frac{(1+c_0)}{2c_0}\, k f, \quad k_1 = \frac{1+3c_0}{2c_0}\, f - \frac{k^2}{2} + gk,$$

$$g_1 = \frac{1+c_0}{2c_0}\, f + \frac{g^2}{2} + \frac{gk}{2}\,. \qquad \text{(V.7.4)}$$

These cannot be solved directly because to be consistent with a Schwarzschild exterior, and to reduce to Newtonian mechanics with the correct $\kappa = 8\pi G/c^4$ we must have g behave like $-\frac{2}{x}$ near $x = 0$, so k behaves like $\frac{x(1+3c_0)f(x)}{6c_0}$ near $x = 0$. The solution with $g(0) = 0$ which is regular at $x = 0$ is not physically acceptable. Of course f is an even function $f(-x) = f(x)$ and g and k are odd. We solve only in the region $x > 0$ and the other half follows by symmetry, the frame component system breaking down at $x = 0$. In order to do numerical integration we must introduce a new function $w = kg$ which is regular at $x = 0$ with $w(0) = -\frac{1}{3}\left(\frac{1+3c_0}{c_0}\right) f(0)$. We can the eliminate g in favour of w to obtain the set of equations

$$f_1 = -\frac{(1+c_0)}{2c_0}\, k f, \quad k_1 = \frac{1+3c_0}{2c_0}\, f - \frac{k^2}{2} + w$$

$$w_1 = w\left[\frac{\frac{3}{2}w + \left(\frac{1+3c_0}{2c_0}\right) f}{k}\right] + \left(\frac{1+c_0}{2c_0}\right) k f\,, \qquad \text{(V.7.5)}$$

where $\delta(0) = \delta_0 > 0$, $k(0) = 0$, $w(0) = -\frac{1}{3}\left(\frac{1 + 3c_0}{c_0}\right)\delta(0)$ are the initial conditions. The quantity in square brackets is taken to be zero at $x = 0$ using l'Hôpital's rule.

The relation (V.7.3) is called the *hydrostatic pressure equation* which reduces, for $c_0 << 1$ to the classical form $\frac{dP}{dx} = \rho a$ where a is acceleration. This equation is used to determine atmospheric pressure at various altitudes.

(V.8) Pressure in an Omnidirectional Universe

It is easy to see that the introduction of pressure $P = c_0(\rho c^2+\epsilon)$ into the omnidirectional cosmology (III.12) with $\hat{f} = 0$ increases the rate of expansion of the universe as one might expect. However, we can show, as is well known from the $k = +1$ Robertson-Walker metric that pressure cannot halt a gravitational collapse. We might have expected this since for an omnidirectional case, the motion is geodesic meaning there is no acceleration.

For $\hat{f} \neq 0$, using the results in (III.12) we see that for $P = c_0(\rho c^2+\epsilon)$ that $\dot{f} = \frac{3c_0 + 1}{4}\hat{f}^2 + \frac{3}{4}(c_0+1)f^2$ and $\dot{\hat{f}} = \frac{1}{2}f\hat{f}$ give us $\dot{\delta} = \frac{3}{2}(c_0+1)f\delta$ where $\delta = \hat{f}^2 + f^2$, i.e. $\rho c^2+\epsilon = \frac{3}{4\kappa}\delta$. This holds whether or not c_0 is constant. However to integrate further, it is convenient to take $c_0 = \frac{1}{3}$, the value for a universe filled only with isotropic radiation (with a "rest" frame in which the isotropy is observed at each point). This gives $T^a{}_a = 0$, what we expect for radiation, namely a trace-free energy tensor. Then $\delta = c_1\hat{f}^4 = f^2 + \hat{f}^2$ where c_1 is a constant, so $\frac{d\tau}{2} = \mp\frac{d\hat{f}}{\hat{f}^2\sqrt{c_1\hat{f}^2-1}}$ which can be integrated to give $\frac{\tau}{2} = \mp\sqrt{c_1 - 1/\hat{f}^2}$. The lowest density is when $\hat{f}^2 = \frac{1}{c_1}$ so $\delta = \frac{1}{c_1}$ occurs at $\tau = 0$ with expansion for $-2\sqrt{c_1} < \tau < 0$ and

contraction for $0 < \tau < 2\sqrt{c_1}$ to infinite density at $\tau = 2\sqrt{c_1}$.

(V.9) No Unidirectional Steady-State Cosmologies

In this section we examine the unidirectional solutions in the case of weakly flow static and directionally invariant $T_j{}^i{}_k$ along all three principal axes $\underset{\sim}{r}$, $\underset{\sim}{s}$ and $\underset{\sim}{t}$. Thus $T_j{}^i{}_k$ are constants with all derivatives zero, and the integrability conditions (III.15.1) are trivially satisfied. The Jacobi identities in (III.16) give us the following:

$$r(f+t) = fs, \quad (f+t)(g-p) = 2r(h-\hat{f}), \quad 2f(g-p) = s(h-\hat{f}),$$

$$2r(g-p) = s(g-p), \quad f(h-\hat{f}) = t(h-\hat{f}), \quad \left(\hat{f} - \frac{q}{2}\right)(f+t) = s\left(p - \frac{k}{2}\right) .$$

If the universe is to be expanding $(f > 0)$ with zero shear $(t = 0)$ as is observed, then we see that $h = \hat{f}$ and $g = p$ which means we are reduced to the case of (III.18), Putting $r(f+t) = fs$ into the expressions for σ_I, $\rho c^2+\epsilon$, and λ we find $\lambda = 0$ (since $t = 0$) so $r = s$ and $\sigma_I = \rho c^2+\epsilon = \frac{3}{4\kappa}(f^2-s^2)$ for $I = 1,2,3$. The pressure here is too great (and of the wrong sign for a fluid) to be consistent with observation. This high "pressure" is needed so that as the universe expands enough work will be done (of PdV type) to add enough energy density ϵ so that as ρc^2 decreases with expansion, $\rho c^2+\epsilon$ remains constant, maintaining the steady state condition. Of course under our steady state hypothesis, both $P = -\sigma_I$, $I = 1,2,3$ and $\rho c^2+\epsilon$ are constants with no time or space derivatives.

As an approximate estimate, if we say the universe is filled with stars that are on the average 90% hydrogen and 10% helium that behave like particles, that all radiation is obtained from H to He conversion which is 1% mass efficient and all pressure is radiation pressure

(= $\frac{1}{3}$ radiation energy density) then we would expect $P \doteq \frac{1}{3} \times 10^{-3}(\rho c^2+\epsilon)$ which is close to a dust solution, for the universe as a whole. For an isotropically expanding universe this rules out the steady state condition we have been considering. As the universe ages, we can show that ultimately the value of $c_0 = \dfrac{P}{\rho c^2+\epsilon}$ decreases, as more mass is converted to radiation and the entropy increases.

(V.10) The Kerr Metric in Frame Components - Interior Solutions

We write the Kerr Metric in Schwarzschild like coordinates as

$$ds^2 = (r^2+a^2\cos^2\theta)\left[\frac{dr^2}{r^2-2mr+a^2} + d\theta^2\right] + \frac{4mar\sin^2\theta}{r^2+a^2\cos^2\theta}\,d\phi dt$$

$$+ \left(r^2 + a^2 + \frac{2mra^2\sin^2\theta}{r^2+a^2\cos^2\theta}\right)\sin^2\theta d\phi^2 - \left(1 - \frac{2mr}{r^2+a^2\cos^2\theta}\right)dt^2.$$

Putting the exact differentials in the form $d^1 = dr$, $d^2 = d\theta$, $d^3 = d\phi$, $d^4 = dt$ and the inexact differentials for a metric frame component system as $\partial^1, \partial^2, \partial^3, \partial^4$ then we have the transformation relation $d^i = v^i_{(b)}\partial^b$ where $v^i_{(b)} = G^{\ \ i}_{\gamma\alpha\ b}$ is (vector) transformation from frame components to homogeneous coordinates. Using $ds^2 = \partial^1\partial^1 + \partial^2\partial^2 + \partial^3\partial^3 - \partial^4\partial^4$ and solving for the $v^i_{(b)}$ we find that the following is a solution:

$$v^1_{(1)} = \sqrt{\frac{r^2-2mr+a^2}{r^2+a^2\cos^2\theta}} = r_{,1}\ , \quad v^2_{(2)} = (r^2+a^2\cos^2\theta)^{-\frac{1}{2}} = \theta_{,2}\ ,$$

$$v^4_{(4)} = \left(1 - \frac{2mr}{r^2+a^2\cos^2\theta}\right)^{-\frac{1}{2}} = t_{,4}\ ,$$

$$v^3_{(3)} = \frac{1}{\sin\theta}\left(\frac{r^2+a^2\cos^2\theta-2mr}{(r^2+a^2\cos^2\theta)(r^2-2mr+a^2)}\right)^{\frac{1}{2}} = \phi_{,3}\ ,$$

$$v^4_{(3)} = \frac{2mar\sin\theta}{\sqrt{(r^2+a^2\cos^2\theta-2mr)(r^2+a^2\cos^2\theta)(r^2-2mr+a^2)}} = t_{,3}\ ,$$

where all others are zero. If we choose d^1, d^2 parallel to ∂^1 and ∂^2 respectively, and insist that $v^3_{(4)} = 0$ in order to get $T_{4AB} = 0$ so that $\theta_{AB} = T_{4(AB)} = 0$ (i.e. rigid motion, or flow stationary condition) then the above values for $v^i_{(b)}$ are uniquely determined.

As in the Schwarzschild case (which this reduces to for $a = 0$) we can get the Ricci coefficients by using the relation $v^i_{(b),c} - v^i_{(c),b} = T_b{}^d{}_c v^i_{(d)}$ where the comma is frame component differentiation. Of course $v^i_{(b),c} = v^i_{(b)\partial j} v^j_{(c)}$ where ∂j is ordinary partial derivative with respect to r, θ, ϕ, t according as whether $j = 1, 2, 3$ or 4. From these results we obtain the Ricci coefficients as

$$T_1{}^1{}_2 = \frac{a^2\cos\theta\,\sin\theta}{(r^2+a^2\cos^2\theta)^{3/2}} = \frac{f}{2}\,, \qquad T_1{}^2{}_2 = \frac{r\sqrt{r^2-2mr+a^2}}{(r^2+a^2\cos^2\theta)^{3/2}} = \frac{s}{2}\,,$$

$$T_1{}^3{}_3 = \frac{r(r^2+a^2\cos^2\theta-2mr)^2 - ma^2\sin^2\theta(r^2-a^2\cos^2\theta)}{(r^2+a^2\cos^2\theta)^{3/2}(r^2+a^2-2mr)^{\frac{1}{2}}(r^2+a^2\cos^2\theta-2mr)} = -\frac{g}{2}\,,$$

$$T_2{}^3{}_3 = \frac{\cot\theta}{\sqrt{r^2+a^2\cos^2\theta}} + \frac{2mra^2\sin\theta\cos\theta}{(r^2+a^2\cos^2\theta)^{3/2}(r^2+a^2\cos^2\theta-2mr)} = -\frac{h}{2}\,,$$

$$T_1{}^4{}_3 = \frac{2ma\sin\theta(r^2-a^2\cos^2\theta)}{(r^2+a^2\cos^2\theta)^{3/2}(r^2+a^2\cos^2\theta-2mr)} = \frac{p}{2}\,,$$

$$T_3{}^4{}_2 = \frac{4mar\cos\theta\sqrt{r^2+a^2-2mr}}{(r^2+a^2\cos^2\theta)^{3/2}(r^2+a^2\cos^2\theta-2mr)} = \frac{q}{2}\,,$$

$$T_1{}^4{}_4 = \frac{m(r^2-a^2\cos^2\theta)\sqrt{r^2-2mr+a^2}}{(r^2+a^2\cos^2\theta)^{3/2}(r^2+a^2\cos^2\theta-2mr)} = \frac{k}{2}\,,$$

$$T_4{}^4{}_2 = \frac{2mra^2\sin\theta\cos\theta}{(r^2+a^2\cos^2\theta)^{3/2}(r^2+a^2\cos^2\theta-2mr)} = -\frac{n}{2}\,,$$

and all others are zero except for natural antisymmetry.

Since $\dot{u}_b = T_b{}^4{}_4$ and $\nu^1 = \frac{1}{2}T_3{}^4{}_2$, $\nu^2 = \frac{1}{2}T_1{}^4{}_3$ we see that for $\theta = 0$ and $\pi/2$ the acceleration is radially directed, and for $\theta = 0$

the vorticity is radial (i.e. along $\underset{\sim}{r}$) and for $\theta = \frac{\pi}{2}$ the vorticity is along $\underset{\sim}{s}$. This is exactly what we would expect for a rotating star. We can now write out the general expression for the Ricci coefficients in the rotating case (here a is an index, not angular momentum) as

$$T_b{}^a{}_c = g t^a t_{[b} r_{c]} + h t^a t_{[b} s_{c]} + k u^a u_{[b} r_{c]} + s s^a r_{[b} s_{c]}$$
$$+ f r^a r_{[b} s_{c]} + p u^a r_{[b} t_{c]} + q u^a t_{[b} s_{c]} + r u^a u_{[b} s_{c]} \qquad \text{(V.10.1)}$$

where $g_{,\ell} = g_1 r_\ell + g_2 s_\ell$, $h_{,\ell} = h_1 r_\ell + h_2 s_\ell$ and similarly for the other functions. These Ricci coefficients are strongly flow-static and weakly directionally invariant (along t^a). They include the Kerr metric as a vacuum solution — the possible exterior solution of a rotating star which we can match to an interior solution for say a rotating perfect fluid in adapted frame components. Rotating galaxies will be considered later.

(V.11) Solution for Rotating Stars (Rigid Motion)

We begin with the basic calculations for the Ricci coefficients (V.10.1) we obtained in the last section. The integrability conditions for the functions here are

$$g_{12} - g_{21} = \frac{f}{2} g_1 + \frac{s}{2} g_2, \quad h_{12} - h_{21} = \frac{f}{2} h_1 + \frac{s}{2} h_2 ,$$
$$k_{12} - k_{21} = \frac{f}{2} k_1 + \frac{s}{2} k_2, \quad s_{12} - s_{21} = \frac{f}{2} s_1 + \frac{s}{2} s_2 , \qquad \text{(V.11.1)}$$

and similarly for f, p, q, r, while all other commutators of frame component derivatives are zero.

The Jacobi identity $T_{[b}{}^a{}_{c,d]} + T_e{}^a{}_{[d} T_b{}^e{}_{c]} = 0$ gives us the conditions

$$g_2 - h_1 = \frac{fg}{2} + \frac{hs}{2} ,$$

$$q_1 + p_2 = \frac{kq}{2} + \frac{hp}{2} + \frac{gq}{2} + \frac{pr}{2} + \frac{pf}{2} - \frac{qs}{2} , \tag{V.11.2}$$

$$k_2 - r_1 = \frac{kf}{2} + \frac{rs}{2} .$$

Working out the Ricci tensor we find there are off diagonal terms $r_{(j}s_{k)}$ and $u_{(j}t_{k)}$ in R_{jk}, and if we are in an adapted frame component system with no heat flow along the $\underset{\sim}{t}$ direction (which is physically very reasonable) then

$$g_2 + h_1 - k_2 - r_1 = gh + kr + \frac{pq}{2} + \frac{sh}{2} + \frac{fk}{2} - \frac{sr}{2} - \frac{fg}{2} ,$$

$$q_2 - p_1 = kp - qr + \frac{ps}{2} + \frac{qf}{2} . \tag{V.11.3}$$

Writing out the principal stresses and mass and energy density we have

$$\sigma_1 = -\frac{1}{2\kappa}\left(r_2 - h_2 + \frac{h^2}{2} + \frac{r^2}{2} + \frac{p^2}{8} - \frac{q^2}{8} + \frac{sk}{2} - \frac{gs}{2} - \frac{gk}{2} - \frac{hr}{2}\right) ,$$

$$\sigma_2 = -\frac{1}{2\kappa}\left(k_1 - g_1 + \frac{g^2}{2} + \frac{k^2}{2} + \frac{q^2}{8} - \frac{p^2}{8} + \frac{hf}{2} - \frac{gk}{2} - \frac{hr}{2} - \frac{fr}{2}\right) ,$$

$$\sigma_3 = -\frac{1}{2\kappa}\left(s_1 + k_1 + r_2 - f_2 + \frac{k^2}{2} + \frac{s^2}{2} + \frac{f^2}{2} + \frac{r^2}{2} + \frac{p^2}{8} + \frac{q^2}{8} + \frac{sk}{2} - \frac{fr}{2}\right) ,$$

$$\rho c^2 + \epsilon = -\frac{1}{2\kappa}\left(s_1 - g_1 - f_2 - h_2 + \frac{g^2}{2} + \frac{s^2}{2} + \frac{f^2}{2} + \frac{h^2}{2} - \frac{3p^2}{8} - \frac{3q^2}{8} + \frac{fh}{2} - \frac{sg}{2}\right) .$$

To solve for a rigidly rotating fluid we impose also the conditions

$$\sigma_1 = \sigma_2 = \sigma_3 = -P \tag{V.11.4}$$

which give two more differential equations. They are

$$r_2 - h_2 - k_1 + g_1 = \frac{g^2}{2} + \frac{k^2}{2} + \frac{q^2}{4} - \frac{p^2}{4} - \frac{r^2}{2} - \frac{h^2}{2} + \frac{hf}{2} + \frac{gs}{2} - \frac{sk}{2} - \frac{fr}{2} \tag{V.11.5}$$

for $\sigma_1 = \sigma_2$ and for $\sigma_2 = \sigma_3$ we have

$$s_1 + r_2 - f_2 + g_1 = \frac{g^2}{2} - \frac{s^2}{2} - \frac{f^2}{2} - \frac{r^2}{2} - \frac{p^2}{4} + \frac{hf}{2} - \frac{gk}{2} - \frac{hr}{2} - \frac{sk}{2} . \tag{V.11.6}$$

Now from (V.11.2) we see that there exist functions Φ and Ψ with $g = 2\Phi_1$, $h = 2\Phi_2$, $k = 2\Psi_1$, $\kappa = 2\Psi_2$, since these equations give the integrability conditions (V.11.1) for Φ and Ψ. We call Φ the *logarithmic potential* and Ψ the *gravitational potential.* In the case of the exterior Kerr solution we find from the expressions for $T_j{}^i{}_k$ in (V.10) that (for b an arbitrary constant),

$$\Psi = -\ln v^4_{(4)} = \frac{1}{2}\ln\left(1 - \frac{2mr}{r^2+a^2\cos^2\theta}\right), \quad (\Psi_{,a} = \dot{u}_a),$$

$$\Phi = \ln\left(bv^3_{(3)}\right) = -\ln\sin\theta + \frac{1}{2}\ln\left(\frac{(r^2+a^2\cos^2\theta-2mr)b^2}{(r^2+a^2\cos^2\theta)(r^2+a^2-2mr)}\right) .$$

Combining (V.11.2) with (V.11.3) we obtain

$$\begin{aligned} g_2 - k_2 &= \frac{gh}{2} + \frac{k\kappa}{2} + \frac{pq}{4} + \frac{sh}{2} - \frac{s\kappa}{2} , \\ h_1 - \kappa_1 &= \frac{gh}{2} + \frac{k\kappa}{2} + \frac{pq}{4} + \frac{fk}{2} - \frac{fg}{2} . \end{aligned} \qquad \text{(V.11.7)}$$

In order to differentiate these expressions, substitute and simplify it is necessary to specify the frame component system in a unique way, free of ambiguity.

(V.12) Change of Frame Components for the Kerr Metric

By examining the equations in the last section (V.11) for a rigidly rotating fluid, we can see there would be a great advantage in solving them to have $\kappa = 0$. We achieve this by transforming the frame component system $(\underset{\sim}{r},\underset{\sim}{s},\underset{\sim}{t},\underset{\sim}{u})$ to the new orthonormal frame $(\underset{\sim}{r}',\underset{\sim}{s}',\underset{\sim}{t}',\underset{\sim}{u}')$ where

$\underset{\sim}{r}' = \cos\gamma\,\underset{\sim}{r} + \sin\gamma\,\underset{\sim}{s}$, $\quad \underset{\sim}{t}' = \underset{\sim}{t}$,

$\underset{\sim}{s}' = -\sin\gamma\,\underset{\sim}{r} + \cos\gamma\,\underset{\sim}{s}$, $\quad \underset{\sim}{u}' = \underset{\sim}{u}$, and γ is a function of r and θ.

Since $\underset{\sim}{u}' = \underset{\sim}{u}$, the new acceleration is $\dot{\underset{\sim}{u}}' = \dot{\underset{\sim}{u}}$ because the metric is

unchanged, and we can choose the rotation function γ so as to make the acceleration parallel to $\underset{\sim}{r}'$. This will also have the advantage of giving the hydrostatic pressure equation (which is a consequence only of $T^{ab}{}_{;b} = 0$ for a perfect fluid) in the simple form (V.7.3). The transformation functions $v^i_{(b)}$ from the coordinates in (V.10) to this new frame component system are the same as those in (V.10) except for the following ones

$$v^1_{(1)} = \sqrt{\frac{r^2-2mr+a^2}{r^2+a^2\cos^2\theta}}\cos\gamma, \qquad v^2_{(2)} = \frac{\cos\gamma}{\sqrt{r^2+a^2\cos^2\theta}},$$

$$v^1_{(2)} = -\sqrt{\frac{r^2-2mr+a^2}{r^2+a^2\cos^2\theta}}\sin\gamma, \qquad v^2_{(1)} = \frac{\sin\gamma}{\sqrt{r^2+a^2\cos^2\theta}},$$

where $\tan\gamma = -\dfrac{2ra^2\sin\theta\cos\theta}{(r^2-a^2\cos^2\theta)\sqrt{r^2-2mr+a^2}}$. In particular the functions Ψ and Φ defined in the last section are unchanged in their definition and formulas. The Ricci coefficients given in (V.10) are now all changed (and much more complicated!) except for the fact that $T_4{}^4{}_2 = 0$. The comma differentiation is also different now.

(V.13) Solution for the Interior by Substitution

Now, with the frame component system in (V.12) we have eliminated the uncertainty of the choice of systems and are ready to solve. For $\mathscr{r} = 0$ in the results of (V.11) we obtain immediately

$$h_1 = \frac{gh}{2} + \frac{pq}{4} + \frac{fk}{2} - \frac{fg}{2}, \qquad k_2 = \frac{kf}{2}$$
$$g_2 = \frac{kf}{2} + \frac{gh}{2} + \frac{pq}{4} + \frac{sh}{2}. \qquad \text{(V.13.1)}$$

If we work out h_{12} and substitute in from (V.11.1) through (V.11.6) expressing everything in terms of 1-derivatives only we finally obtain

$$k\left(g_1+\frac{q^2}{8}-\frac{g^2}{2}+\frac{3p^2}{8}+\frac{sk}{2}-\frac{fh}{2}\right) = \left(h_2+\frac{q^2}{8}-\frac{p^2}{8}-\frac{h^2}{2}+\frac{sg}{2}-\frac{sk}{2}+\frac{gk}{2}\right)_{,1}$$

and putting $P = \frac{\delta}{2\kappa} = -\sigma_1$ we see the right hand side is simply $-\delta_{,1}$, and the k on the left hand side tells us this is simply the hydrostatic pressure equation, since k is the acceleration.

If we take the equation $g_2 - k_2 = \frac{h}{2}(g+s) + \frac{pq}{4}$ and take 1-derivatives, then use (V.11.1) through (V.11.6) to reduce it down to an expression involving 2-derivatives only we find that

$$0 = \left(g_1 - k_1 - \frac{fh}{2} - \frac{q^2}{8} + \frac{p^2}{8} - \frac{g^2}{2} + \frac{gk}{2} - \frac{k^2}{2}\right)_{,2}$$

and comparing with the expression for σ_2 we see that $\delta_{,2} = 0$. This is a most remarkable result. It states that the pressure is a function only of the gravitational potential Ψ since it is a constant on the surfaces of constant Ψ.

Next we consider the equation of state $P = c_0(\rho c^2+\epsilon)$ for $0 < c_0 < 1$ where c_0 is a function of temperature or other thermodynamic variables. The hydrostatic pressure equation is given by (V.7.3) and this implies that

$$\begin{aligned}\delta &= \frac{2c_0}{1+c_0}\left(g_1+\frac{q^2}{8}-\frac{g^2}{2}+\frac{3p^2}{8}+\frac{sk}{2}-\frac{fh}{2}\right), \\ &= -h_2-\frac{q^2}{8}+\frac{p^2}{8}+\frac{h^2}{2}-\frac{sg}{2}+\frac{sk}{2}-\frac{gk}{2}.\end{aligned} \qquad \text{(V.13.2)}$$

It should be remarked that we can do this simply because the hydrostatic pressure equation can be derived in any space-time by examining $\dot{u}_a T^{ab}{}_{;b} = 0$ where $T^{ab} = -(\rho c^2+\epsilon)u^a u^b - P\gamma^{ab}$ is the energy tensor for a perfect fluid, whether or not Einstein's equations are assumed. Of course we could also write out $P = c_0(\rho c^2+\epsilon)$ directly using the results of (V.11). This gives

$$g_1 + f_2 + \left(1 + \frac{1}{c_0}\right) h_2 - s_1 = \frac{g^2}{2} + \frac{s^2}{2} + \frac{f^2}{2} + \left(1 + \frac{1}{c_0}\right)\frac{h^2}{2}$$

$$- \left(3 - \frac{1}{c_0}\right) \frac{p^2}{8} - \left(3 + \frac{1}{c_0}\right) \frac{q^2}{8} - \left(1 + \frac{1}{c_0}\right) \frac{sg}{2} + \frac{fh}{2} + \frac{sk}{2c_0} - \frac{gk}{2c_0} . \tag{V.13.3}$$

Combining this with (V.11.6) to eliminate $s_1 - f_2$ we obtain (V.13.2) as a check of accuracy. This incidentally could be used directly to obtain the hydrostatic pressure equation simply from the results in (V.11) since we know that Einstein's equations imply $T^{ab}{}_{;b} = 0$.

The integrability conditions for $g - k$ and h have already been imposed, so the next logical step is to examine the integrability condition for k. Using $\sigma_2 = -\delta/2\kappa$ and (V.13.2) we see that

$$k_1 = \frac{3}{2}\,\delta + \frac{\delta}{2c_0} - \frac{q^2}{4} - \frac{p^2}{4} - \frac{sk}{2} + \frac{gk}{2} - \frac{k^2}{2} , \tag{V.13.4}$$

and of course also

$$g_1 = \frac{1}{2}\,\delta + \frac{\delta}{2c_0} - \frac{q^2}{8} + \frac{g^2}{2} - \frac{3p^2}{8} - \frac{sk}{2} + \frac{fh}{2} . \tag{V.13.5}$$

These equations should be compared to the Schwarzschild ones in (V.7.4) remembering that there, $f = p = q = 0$ and $g = -s$.

If we write $c_{02} = (c_0)_{,2}$ the integrability condition for k gives us the equation

$$k(f_1 + s_2) = \frac{k^2 f}{2} + \frac{kgh}{2} + \frac{ksh}{2} + pq_1 - qp_1 - \frac{5}{4}\,pkq - \frac{hp^2}{2} - \frac{pgq}{2}$$

$$- \left(3 + \frac{1}{c_0}\right) f\delta - \delta\,\frac{c_{02}}{2 c_0^2} - \frac{fgk}{2} , \tag{V.13.6}$$

using (V.13.1), (V.13.4) and (V.11.1) through (V.11.3). Also, from (V.13.5) and (V.11.6) we obtain

$$s_1 - f_2 = -\frac{1}{2}\,\delta - \frac{\delta}{2c_0} + \frac{q^2}{8} + \frac{p^2}{8} - \frac{s^2}{2} - \frac{f^2}{2} - \frac{gk}{2} . \tag{V.13.7}$$

Notice that the pair of equations (6) and (7) have a similar character

to the equations in (V.11.2) and (V.11.3) for $q_1 + p_2$ and $q_2 - p_1$. Imposing the integrability conditions for q, p, s and f on these four equations, is the final stage of solving the problem. Just for completeness, we record the remainder of the derivatives here:

$$h_2 = -\mathfrak{z} - \frac{q^2}{8} + \frac{p^2}{8} + \frac{h^2}{2} - \frac{sg}{2} + \frac{sk}{2} - \frac{gk}{2} ,$$

$$\mathfrak{z}_2 = 0, \tag{V.13.8}$$

$$\mathfrak{z}_1 = - \frac{(1+c_0)}{2c_0} k\mathfrak{z} .$$

Now imposing the integrability condition for $\mathfrak{z}$ using (8) we see that $c_{02} = 0$ so not only P but also $\rho c^2 + \epsilon$ is a function only of Ψ the gravitational potential. Hence the c_{02} term can be dropped in (V.13.6) above. Since $\Psi_1 = \frac{k}{2}$, $\Psi_2 = 0$, $(\ell n\, \mathfrak{z})_1 = -\left(\frac{1+c_0}{c_0}\right) \Psi_1$ or $\frac{d(\ell n\, \mathfrak{z})}{d\Psi} = -\left(1 + \frac{1}{c_0}\right)$. Remember Ψ is negative and increases to zero as we head from the centre of our star out to infinity. Since c_0 is also a function of Ψ, if it is known, we can integrate out from the interior of the star to obtain $\mathfrak{z}$ for each value of Ψ.

(V.14) Structure Identity for the Geometry Structure Scalars

If we have a function ϕ satisfying the integrability conditions $\phi_{,ab} - \phi_{,ba} = T_{a\ b}^{\ c}\phi_{,c}$, then we can differentiate to get $\phi_{,ab\ell} - \phi_{,ba\ell} = T_{a\ b,\ell}^{\ c}\phi_{,c} + T_{a\ b}^{\ c}\phi_{,c\ell}$ and using the integrability condition $\phi_{,\ell ab} - \phi_{,\ell ba} = T_{a\ b}^{\ c}\phi_{,\ell c}$ for $\phi_{,\ell}$ we obtain

$$\begin{aligned} \phi_{,ab\ell} - \phi_{,ba\ell} - \phi_{,\ell ab} + \phi_{,\ell ba} &= (T_{a\ b,\ell}^{\ c} + T_{e\ \ell}^{\ c}T_{a\ b}^{\ e})\phi_{,c} \\ &= -S_{a\ b\ell}^{\ c}\phi_{,c} \end{aligned} \tag{V.14.1}$$

where $S_{a\ b\ell}^{\ c}$ are the geometry structure scalars of (III.7). We call (V.14.1) the *structure identity* for $S_{a\ b\ell}^{\ c}$. It holds for any function ϕ.

Since ϕ is arbitrary, antisymmetrizing in a, b, ℓ gives us the Jacobi identity.

This identity is interesting in relation to the problem of rigid rotation considered in (V.10) through (V.13). Taking $\phi = \Psi$, the gravitational potential, gives us the integrability condition for k, i.e. (V.13.6), and $\phi = \Phi$ the logarithmic potential, gives us the integrability conditions for g and h we had earlier.

(V.15) <u>The Vorticity Potential</u>

If Ψ is the gravitational potential $\Psi_1 = \frac{k}{2}$, $\Psi_2 = 0$ and Φ is the logarithmic potential, $\Phi_1 = \frac{g}{2}$, $\Phi_2 = \frac{h}{2}$, and if we put $\alpha = e^{-\Phi-\Psi}$ and $\beta = e^{2\Psi}$ we see that using (V.11.2) and (V.11.3) to obtain $\alpha(q_1+p_2) + \beta(q_2-p_1)$ it follows that

$$(\alpha p+\beta q)_2 - (\beta p-\alpha q)_1 = \frac{f}{2}(\alpha p+\beta q) + \frac{s}{2}(\beta p-\alpha q).$$

This statement is simply an integrability condition, so there exists a function Λ called the *vorticity potential* with $\Lambda_1 = \alpha p + \beta q$ and $\Lambda_2 = \beta p - \alpha q$. (The components of the vorticity are $v^1 = \frac{q}{4}$, $v^2 = \frac{p}{4}$, $v^3 = 0$.) The functions α and β satisfy the conditions

$$\alpha_1 = -\alpha\left(\frac{g}{2}+\frac{k}{2}\right), \qquad \alpha_2 = -\alpha\,\frac{h}{2},$$
$$\beta_1 = \beta k \qquad , \qquad \beta_2 = 0\,. \tag{V.15.1}$$

Furthermore we can show that

$$p = \frac{\alpha\Lambda_1 + \beta\Lambda_2}{\alpha^2+\beta^2}, \qquad q = \frac{\beta\Lambda_1 - \alpha\Lambda_2}{\alpha^2+\beta^2}, \tag{V.15.2}$$

$p^2+q^2 = \dfrac{\Lambda_1^2+\Lambda_2^2}{\alpha^2+\beta^2}$, so p and q are expressed in terms of functions and derivatives of functions whose integrability conditions are known to be satisfied.

(V.16) Classification of Ricci Coefficients

Although the solution is not complete (integrability conditions for $\mathfrak{s}$ and f have not been imposed on (V.13.6) and (V.13.7)) we have enough information already to detail the interior structure of a rigidly rotating perfect fluid to quite an extent. The Ricci coefficients (V.10.1) with $\mathfrak{r} = 0$ and the other 7 functions having 1 and 2-derivatives only, describe the adapted frame component systems in a class of space-times that we say are of *Kerr type for rigid motion which are radially adapted.* The general Kerr type Ricci coefficients are given by (all functions have 1 and 2-derivatives only)

$$T_b{}^a{}_c = gt^a t_{[b}r_{c]} + ht^a t_{[b}s_{c]} + \mathfrak{k}u^a u_{[b}r_{c]} + \mathfrak{s}s^a r_{[b}s_{c]}$$

$$+ fr^a r_{[b}s_{c]} + \mathfrak{p}u^a r_{[b}t_{c]} + \mathfrak{q}u^a t_{[b}s_{c]} + \mathfrak{r}u^a u_{[b}s_{c]}$$

$$+ vt^a u_{[b}r_{c]} + wt^a u_{[b}s_{c]}. \qquad \text{(V.16.1)}$$

For these, we have the acceleration is $\dot{u}_a = T_a{}^4{}_4 = \frac{\mathfrak{k}}{2} r_a + \frac{\mathfrak{r}}{2} s_a$, the vorticity is $\upsilon^a = \frac{\mathfrak{q}}{4} r^a + \frac{\mathfrak{p}}{4} s^a$ and the deformation rate is $\theta_{AB} = T_{4(AB)} = -\frac{v}{2} t_{(A}r_{B)} - \frac{w}{2} t_{(A}s_{B)}$. These Ricci rotation coefficients are weakly flow-static, directionally invariant, and represent an isochoric shearing. They are not flow-stationary unless $v = w = 0$ in which case the motion is (strongly) flow-static and is therefore rigid. If $\mathfrak{r} = 0$ we say they (the Ricci coefficients) are *radially adapted* to the acceleration. The $T_b{}^a{}_c$ in (V.16.1) are the most general Ricci coefficients one could obtain from the Kerr metric in (V.10) using a flow vector derived from the $\phi_{,\ell}$ and $t_{,\ell}$ directions, so as to give the weak flow-static, directionally invariant condition. It should be remarked that the exterior may not be specifically the Kerr

solution. Herlt[1] claims a rigidly rotating perfect fluid cannot have a Kerr exterior, but Roos[2] does not agree.

For a viscous fluid undergoing shearing, we could solve using (V.16.1) with $\kappa = 0$ (radially adapted) and introduce a difference in the principle stresses σ_1, σ_2, σ_3 proportional to the shearing terms v and w and a coefficient of viscosity. For rotating galaxies we could take (V.16.1) and put $\kappa = k = 0$ (zero acceleration), $P = 0$, (dust), and $q = 0$. Thus we orient the frame component system along the vorticity, so $\nu^a = \frac{p}{4} s^a$ and the r^a direction is truly the radial direction in a cylindrical sense. The existence of such solutions is not known now, or whether the introduction of pressure is necessary. The Kerr type Ricci coefficients cover all stationary axisymmetric metrics of the form $ds^2 = Adr^2 + Bd\theta^2 + Cdrd\theta + Dd\phi^2 + Edt^2 + Fd\phi dt$ where A, B, C, D, E, F are functions of r and θ. The only transformation coefficients which can be non-zero are $v^1_{(1)}$, $v^2_{(2)}$, $v^1_{(2)}$, $v^2_{(1)}$, $v^3_{(3)}$, $v^4_{(4)}$, $v^3_{(4)}$, $v^4_{(3)}$.

Let us consider a few extra results here. One can show, using the formula for the exterior derivative in (III.6) plus the integrability conditions and Jacobi identity that the following is true.

<u>Derivative Contraction Theorem</u>: *The 2-form whose value in frame components is* $T_j{}^i{}_{k,i}$ *is closed, i.e.* $dT_j{}^i{}_{k,i}\big|_\ell = 0$.

In the special case considered in (V.13) we can find a 1-form whose exterior derivative is this 2-form in a very simple way, since

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[1] Herlt, E., Ann. der Physik <u>24</u>, 177-187 (1970).

[2] Roos, W., G.R.G. <u>7</u>, 431-444 (1976).

$\left(\frac{f_1+\delta_2}{2}\right)(r_a s_b - s_a r_b) = d\left(\frac{\delta}{2} r_a - \frac{f}{2} s_a\right)\Big|_b$. We should also remark that the vorticity potential Λ for the Kerr metric is explicitly known to be

$$\Lambda = -\frac{4mra\sin^2\theta}{b(r^2+a^2\cos^2\theta-2mr)} - \frac{4ma\cos\theta}{r^2+a^2\cos^2\theta} \qquad (V.16.1)$$

where b is an arbitrary constant with dimensions of length that is the same b as found in the expression for the logarithmic potential Φ in (V.11). This is for the frame component system in (V.10) where $\hbar \neq 0$, for which Λ can be defined analogously to (V.15).

(V.17) Elastic Unidirectional Solids — Exact Omnidirectional Solutions

In this section we will examine specific motion types that are permissible for the special constitutive equation of linear elasticity, namely $\hat{T}^{ab} = E^{abcd}(\gamma_{cd}-\hat{\gamma}_{cd})$. We assume that the material derivatives $\hat{\gamma}_{cd\wedge e}$ and $E^{abcd}{}_{\wedge e}$ are zero, and the space-time and body manifolds are unidirectional (V.2). Furthermore, we will take $h = \hat{f}$ and $g = p$ in (III.15) in order to satisfy (V.3.1) and as a consequence, we can use the results of (III.18). Since E^{abcd} and $\hat{\gamma}_{cd}$ are materially constant, they project to proper time independent tensors $E^{\alpha\beta\gamma\delta}$ and $\hat{\gamma}_{\alpha\beta}$ on B, which have zero material derivative with respect to the material connection $\Gamma_{\beta}{}^{\alpha}{}_{\gamma}$ on B. But the components of $\Gamma_{\beta}{}^{\alpha}{}_{\gamma}$ are zero in frame components on B (V.1), so $E^{\alpha\beta\gamma\delta}$ and $\hat{\gamma}_{\alpha\beta}$ are constants, independent of $X \in B$, when evaluated in the unidirectional frame component system. We assume the symmetries $E^{abcd} = E^{cdab} = E^{bacd} = E^{abdc}$ are valid, so that we may write as one solution, not the most general,

$$E^{\alpha\beta\gamma\delta} = \mu X^\alpha X^\beta X^\gamma X^\delta + \nu(Y^\alpha Y^\beta + Z^\alpha Z^\beta)(Y^\gamma Y^\delta + Z^\gamma Z^\delta)$$
$$+ \sigma[X^\alpha X^\beta(Y^\gamma Y^\delta + Z^\gamma Z^\delta) + (Y^\alpha Y^\beta + Z^\alpha Z^\beta)X^\gamma X^\delta]$$

where μ, ν, σ are constants, which are moduli of elasticity, and $X^\alpha = \delta^\alpha_1$, $Y^\alpha = \delta^\alpha_2$, $Z^\alpha = \delta^\alpha_3$. This incorporates the unidirectional symmetry. Furthermore, we will take, for ρ_0 constant, $\hat{\gamma}_{\alpha\beta} = \rho_0^{-2/3}\delta_{\alpha\beta}$ as the relaxation local metric on the body. In this system, on the body we have the constitutive equation $\hat{T}^{\alpha\beta} = P^\alpha_a P^\beta_b \hat{T}^{ab} = E^{\alpha\beta\gamma\delta}(g_{\gamma\delta}(\tau)-\hat{\gamma}_{\gamma\delta})$ where $\hat{T}^{ab} = \sigma_1 r^a r^b + \sigma_2 s^a s^b + \sigma_3 t^a t^b$, $g_{\gamma\delta}(\tau) = \gamma_{cd}P^c_\gamma P^d_\delta$, and $r^a = \delta^a_1$, $s^a = \delta^a_2$, $t^a = \delta^a_3$, $a,b = 1,2,3,4$. If we substitute, using the value of P^α_a given in (V.2) namely

$$P^\alpha_a = aX^\alpha r_a + b\cos\theta(Y^\alpha s_a + Z^\alpha t_a) + b\sin\theta(Y^\alpha t_a - Z^\alpha s_a),$$

$$P^a_\alpha = \frac{1}{a} r^a X_\alpha + \frac{1}{b}\cos\theta(s^a Y_\alpha + t^a Z_\alpha) + \frac{1}{b}\sin\theta(t^a Y_\alpha - s^a Z_\alpha),$$

we find that $g_{\gamma\delta}(\tau) = \frac{1}{a^2} X_\gamma X_\delta + \frac{1}{b^2}(Y_\gamma Y_\delta + Z_\gamma Z_\delta)$ and using $\delta_{\gamma\delta} = X_\gamma X_\delta + Y_\gamma Y_\delta + Z_\gamma Z_\delta$ we have finally that

$$\hat{T}^{\alpha\beta} = [\mu(a^{-2}-\rho_0^{-2/3})+2\sigma(b^{-2}-\rho_0^{-2/3})]X^\alpha X^\beta + [\sigma(a^{-2}-\rho_0^{-2/3})+2\nu(b^{-2}-\rho_0^{-2/3})](Y^\alpha Y^\beta + Z^\alpha Z^\beta)$$

and hence the principal stresses are given by

$$\sigma_1 = \frac{1}{a^2}[\mu(a^{-2}-\rho_0^{-2/3})+2\sigma(b^{-2}-\rho_0^{-2/3})],\quad \sigma_2 = \sigma_3 = \frac{1}{b^2}[\sigma(a^{-2}-\rho_0^{-2/3})+2\nu(b^{-2}-\rho_0^{-2/3})].$$

(The density $\rho = ab^2$ is measured relative to $\rho = \rho_0$ in the relaxation configuration of zero stress.) These are to be compared with the values in (III.18), namely

$$\sigma_1 = -\frac{1}{2\kappa}\left(2\dot{f} + \frac{r^2}{2} + rs - \frac{3f^2}{2}\right),\quad \sigma_2 = \sigma_3 = -\frac{1}{2\kappa}\left(2\dot{f} - r' - s' + \dot{t} - \frac{3f^2}{2} + \frac{r^2}{2} - \frac{3ft}{2} + \frac{s^2}{2} - \frac{t^2}{2} + \frac{rs}{2}\right).$$

Of course all functions must be integrable with integrability conditions like f, r, s, t satisfy in (III.18). Also the Jacobi identities of (III.18) must be satisfied, as well as the equations (V.2.2). These

latter equations imply $w = v$ (for the Ricci coefficients on the body), $\dot{a} = \frac{a}{2}(f+t)$, $b' = \frac{r}{2}b$ and $\dot{b} = \frac{f}{2}b$. The Jacobi identity $f' - \dot{r} + \frac{rf}{2} + \frac{tr}{2} - \frac{fs}{2} = 0$ is merely the integrability condition for b, and we have also $\rho c^2 + \epsilon = -\frac{1}{2\kappa}\left(-2r' - \frac{3f^2}{2} + \frac{3r^2}{2} - ft\right) = ab^2c^2 + \epsilon$ and the heat flux is $\lambda = \frac{1}{\kappa}\left(f' + \frac{tr}{2}\right)$.

This, then, formulates the problem for a unidirectional linearly elastic solid. Let us solve specifically in the omnidirectional case. Here $E^{\alpha\beta\gamma\delta} = \mu\delta^{\alpha\beta}\delta^{\gamma\delta}$ so $\mu = \nu = \sigma$, and r, s, t are zero, with only f non-zero and f possessing only a proper time derivative, i.e. $f' = 0$, $\dot{f} = -\frac{df}{d\tau}$. Then $a = b$, $a' = 0$, $\dot{a} = \frac{a}{2}f$ and $3\mu a^{-2}(a^{-2} - \rho_0^{-2/3}) = -\frac{1}{2\kappa}\left(2\dot{f} - \frac{3f^2}{2}\right)$ where μ and ρ_0 are constants. For the Ricci coefficients on the body we have $w = 0$ by omnidirectionality (V.1) and in this case (where $\hat{f} = 0$) we also have $v = 0$, otherwise $av = \hat{f}$ by (V.2.2) and (V.3.1).

The differential equation for a is (for $\hat{f} = 0$), $2a\ddot{a} - 5\dot{a}^2 = -3\kappa\mu(a^{-2} - \rho_0^{-2/3})$ where κ, μ and ρ_0 are constants. If we look for a solution of the form $\dot{a} = \pm(c_1a^{-2} - c_2 + c_3a^5)^{1/2}$ where c_1, c_2, c_3 are constants we find $c_1 = \frac{3}{7}\kappa\mu$, $c_2 = \frac{3}{5}\kappa\mu\rho_0^{-2/3}$ and c_3 is undetermined. Of course $\rho = a^3$ and therefore $a > 0$ always. We cannot have $c_3 = 0$ otherwise the condition $\rho \le \left(\frac{5}{7}\right)^{3/2}\rho_0$ would mean the principal stresses (a negative "pressure" here allowed for solids but not fluids) would be greater in magnitude than the mass and energy density. We must have $c_3 > 0$ to keep the quantity (whose square root is being taken to determine $\dot{a}$) positive for high density. Now $f = \pm 2\sqrt{c_1a^{-4} - c_2a^{-2} + c_3a^3}$ and $\dot{f} = -4c_1a^{-4} + 2c_2a^{-2} + 3c_3a^3$. The inequality that states that the magnitude of the stress does not exceed mass and energy density is $\frac{3f^2}{2} \ge \dot{f} \ge 0$ which can be written as

$$6c_1a^{-4} - 6c_2a^{-2} + 6c_3a^3 \geq -4c_1a^{-4} + 2c_2a^{-2} + 3c_3a^3 \geq 0, \qquad \text{(V.17.1)}$$

which is valid for high densities, but not for very low densities. Of course $\epsilon = \rho c^2 + \epsilon - c^2a^3$ where c is the speed of light, so the internal energy density is given by

$$\epsilon = \frac{3c_1}{\kappa}a^{-4} - \frac{3c_2}{\kappa}a^{-2} + \left(\frac{3c_3}{\kappa} - c^2\right)a^3 \qquad \text{(V.17.2)}$$

which must always be positive in the range of physical interest for the solution. Hence $\dot{a}^2 \geq \frac{\kappa c^2}{3}a^5$. Because of this condition we find an infinite density singularity and a low density breakdown with finite time separation, the system moving toward the former if $\dot{a} < 0$ and the latter if $\dot{a} > 0$. This is true since $\frac{da}{d\tau} = -\dot{a}$. The low density breakdown is an indication of the failure of the naive linear elasticity approximation to be valid for a very large stretching. Of course the omnidirectional approximation is a physical oversimplification that is of more interest in cosmology than elasticity. For stars, we would do better with the spherical Ricci coefficients (III.26.1) or the Kerr type (V.16.1), but we have selected this simple form in order to be able to find explicit solutions to the differential equations that can be analyzed to show how the description of the motion of a body in space time can be applied. For simplicity we have also ignored thermodynamic effects on the value of μ.

If $\hat{f} \neq 0$, then $\hat{f} = va$ where v is a constant, and $T_{\beta\ \gamma}^{\ \alpha} = v\epsilon_{\beta\ \gamma}^{\ \alpha}$ are the Ricci coefficients on the body (V.1). Here the differential equation is $2a\ddot{a} - 5\dot{a}^2 = \frac{v^2a^4}{4} - 3\kappa\mu(a^{-2} - \rho_0^{-2/3})$ with solution $\dot{a} = \pm\sqrt{c_1a^{-2} - c_2 + c_3a^5 - c_4a^4}$ where $c_4 = \frac{v^2}{4}$ and c_1, c_2, c_3 are as before. Also we have $f = \pm 2\sqrt{c_1a^{-4} - c_2a^{-2} + c_3a^3 - c_4a^2}$ and $\dot{f} = -4c_1a^{-4} + 2c_2a^{-2} + 3c_3a^3 - 2c_4a^2$. The identity that states that stress

is less than mass and energy density is again (V.17.1) without change, and the internal energy density is given by (V.17.2), the c_4 terms cancelling out. Of course this time $\sigma_1 = -\frac{1}{2\kappa}\left(2\dot{f} - \frac{3}{2} f^2 - \frac{1}{2} \hat{f}^2\right)$ and $\rho c^2 + \epsilon = \frac{3}{4\kappa}(f^2+\hat{f}^2)$. Here we may find a strongly flow-static solution with $\rho c^2 + \epsilon = 3\sigma_1 = -3P$ for negative "pressure" corresponding to the elastic material being stretched to a greater volume and lower density than ρ_0. The conditions are that a be constant, f and $\dot{f}$ be zero, and hence c_4 and c_3 must be such that

$$c_1 a^{-2} + c_3 a^5 = c_2 + c_4 a^4$$

and

$$4c_1 a^{-2} + 2c_4 a^4 = 2c_2 + 3c_3 a^5$$

hold for some $a > 0$. Also for this a we require that $\epsilon = \frac{3}{4\kappa} v^2 a^2 - c^2 a^3 \geq 0$. Since the argument of a square root must be non-negative, this value of a must be a relative minimum for $c_1 a^{-2} - c_2 + c_3 a^5 - c_4 a^4$ that reduces its value to zero. The stability of the equilibrium depends on the choice of sign in the expression for $\dot{a}$.

(V.18) Static and Stationary Space-Times

In this section we take these familiar definitions in relativity and apply them to frame components. We are interested only in those symmetry conditions where the time-like Killing vector is parallel to the matter flow, and will assume this from now on when referring to stationary or static.

The condition $L_{\phi \underset{\sim}{u}} g_{ab} = 0$ for some scalar field ϕ is equivalent to $\theta_{ab} = 0$ and $\dot{u}_a = \Psi_{,a}$ where $\Psi = \ell n\, \phi$. In frame components $\theta_{ab} = 0$ means $T_{4(AB)} = 0$ and $d\dot{u}_a|_b = 0$ means $\dot{u}_{a,b} - \dot{u}_{b,a} - \dot{u}_c T_a{}^c{}_b = 0$, or putting $\dot{u}_c = T_c{}^4{}_4$ we have

$$T_{a\ 4,b}^{\ 4} - T_{b\ 4,a}^{\ 4} = T_{c\ 4}^{\ 4} T_{a\ b}^{\ c} \,. \tag{V.18.1}$$

These are the conditions required for the space-time (with the material flow) to be stationary. In particular this implies the condition of flow-stationary $(\dot{\theta}_{ab} = 0)$. We see the Kerr-type Ricci coefficients for rigid motion describe a stationary case (with Killing vector parallel to the flow) since we proved in (V.11) the existence of a gravitational potential Ψ. We will examine the unidirectional stationary solutions later.

In the static case we write the metric in (homogeneous) coordinates as $ds^2 = -e^{2\Psi}dt^2 + \gamma_{ij}dx^i dx^j$ where $\gamma_{ij\natural 4} = 0 = \Psi_{\natural 4}$ and $\natural$ is coordinate partial differentiation. Ψ is the gravitational potential, $g_{44} = -e^{2\Psi}$, $g_{4J} = 0$ and $g_{IJ} = \gamma_{IJ}$, $I,J = 1,2,3$. The flow-vector in coordinates is

$$u^4 = e^{-\Psi}, \quad u^1 = u^2 = u^3 = 0,$$
$$u_4 = -e^{\Psi}, \quad u_1 = u_2 = u_3 = 0, \quad u^a u_a = -1,$$

and $e^{\Psi}\underset{\sim}{u}$ is a time-like Killing vector. Evaluating $\left\{ {}_{j}{}^{i}{}_{k} \right\}$ for $i = 4$ gives $\left\{ {}_{I}{}^{4}{}_{J} \right\} = 0 = \left\{ {}_{4}{}^{4}{}_{4} \right\}$ and $\left\{ {}_{4}{}^{4}{}_{I} \right\} = \left\{ {}_{I}{}^{4}{}_{4} \right\} = \Psi_{\natural I}$. Working out $u_{a;b} = u_{a\natural b} - u_c\left\{ {}_{a}{}^{c}{}_{b} \right\} = \theta_{ab} + \omega_{ab} - \dot{u}_a u_b$ gives $\theta_{ab} = \omega_{ab} = 0$ and $\dot{u}_I = \Psi_{\natural I}$. Hence we make the following definition.

The Ricci coefficients $T_{j\ k}^{\ i}$ are said to be *stationary* if $T_{j\ k}^{\ i}$ satisfy (V.18.1) and $T_{4(AB)} = 0$. They are *static* if they are stationary and also $T_{A\ B}^{\ 4} = 0$, $A,B = 1,2,3$. Thus we have set $\omega_{ab} = 0$ as well as $\theta_{ab} = 0$, while, of course, $d\dot{u}_a\big|_b = 0$. It is easy to see that in the static case, the condition $T_{j\ k,4}^{\ i} = 0$ can be assumed, by appropriate choice of metric frame components in transforming from the coordinates above. Now we demonstrate a remarkable theorem showing that the general stationary metric has strongly flow static Ricci

coefficients for an adapted frame component system. We can also prove the converse so that we may say *strongly flow-static is equivalent to stationary*.

We write the general stationary metric in the form $ds^2 = -e^{2\Psi}dt^2 + g_{4J}dt\,dx^J + \gamma_{ij}dx^i dx^j = g_{ij}dx^i dx^j$ where $\gamma_{4j} = 0$, $\Psi_{\varsigma 4} = g_{4J\varsigma 4} = \gamma_{ij\varsigma 4} = g_{ij\varsigma 4} = 0$, $g_{44} = -e^{2\Psi}$, $x^4 = t$ and $e^{\Psi}\underset{\sim}{u}$ is a Killing vector, where $u^4 = e^{-\Psi}$, $u^1 = u^2 = u^3 = 0$, $u_4 = -e^{\Psi}$, $u_J = g_{4J}e^{-\Psi}$, $J = 1,2,3$, and $\theta_{ij} = 0$, $\Psi_{\varsigma i} = \dot{u}_i$ in these coordinates. The 4-flow vector u^i is thus determined and $u^i u_i = -1$. The objective is to find Ricci coefficients with $T_4{}^A{}_B = 0$ and $T_b{}^a{}_{c,4} = 0$ (i.e. [strongly] flow static). Writing $ds^2 = \partial^1\partial^1 + \partial^2\partial^2 + \partial^3\partial^3 - \partial^4\partial^4$ with $dx^i = v^i_{(a)}\partial^a$ we can find $v^i_{(a)}$ with $v^i_{(a)\varsigma 4} = 0$ and $v^1_{(4)} = v^2_{(4)} = v^3_{(4)} = 0$. We begin with $v^4_{(4)} = e^{-\Psi}$ so $-e^{2\Psi}dt^2 = -\partial^4\partial^4$ and continue to solve for the other $v^i_{(a)}$ from $g_{4J}dx^4dx^J + \gamma_{IJ}dx^Idx^J = \partial^1\partial^1 + \partial^2\partial^2 + \partial^3\partial^3$. As a result using $v^i_{(a),b} - v^i_{(b),a} = T_a{}^c{}_b v^i_{(c)}$ we see that $T_4{}^A{}_B = 0$ and $T_c{}^a{}_{b\varsigma 4} = 0$. But for any function ϕ, $\phi_{,4} = \phi_{\varsigma 4}v^4_{(4)}$ so that $v^i_{(a),4} = 0 = T_b{}^a{}_{c,4}$. Hence these Ricci coefficients are flow-static. Furthermore, we can rotate the 3-frame orthogonal to u^a without affecting the condition $v^J_{(4)} = 0$ so as to put us in adapted frame components, with the orthogonal part of the Ricci tensor diagonal.

Conversely, suppose the Ricci coefficients are flow-static, i.e. $T_4{}^A{}_B = 0$ and $T_b{}^a{}_{c,4} = 0$. In the light of the Jacobi identity $T_{[a}{}^4{}_{b,4]} + T_{c[4}{}^4 T_a{}^c{}_{b]} = 0$ we see that (V.18.1) is equivalent to

$$T_a{}^4{}_{b,4} + T_A{}^4{}_a T_b{}^A{}_4 + T_A{}^4{}_b T_4{}^A{}_a = 0.$$

In the flow-static case, $T_a{}^4{}_{b,4} = 0$, and we can quickly check for $a,b = 4$ and I or I and J in the different combinations that the other terms add to zero using $T_4{}^A{}_b = 0$. Since $\theta_{ab} = 0$ is obvious,

the conditions for the Ricci coefficients to be stationary by definition are satisfied, so a time-like Killing vector parallel to the flow exists.

We note that in the flow-static case, $T_{b\ c,d4}^{\ a} = 0$ also, so $R^{a}_{\ bcd,4} = 0$ for the Riemann tensor in particular.

Of course we may have non-rigid motion, even though a time-like Killing vector exists which is not parallel to the flow. Such would be the case if we took the Kerr metric with a different flow vector than the one used in (V.10) and ended up with Kerr type Ricci coefficients (V.16.1) with shearing. In that case, they would only be weakly directionally invariant and (weakly) flow-static, an indication of the non-parallel Killing vector.

It should be remarked that just from the definition of stationary Ricci coefficients alone (namely $T_{4(AB)} = 0$ and (V.18.1) holds) we cannot prove the flow-static conditions for $T_{b\ c}^{\ a}$ are satisfied, since the orthogonal part of the frame defining $T_{b\ c}^{\ a}$ may contain time dependent rotations. In the flow-static case these have been transformed away.

(V.19) <u>Stationary Unidirectional Cases (Relativistic Ideal Gas)</u>

From (III.21) we see that in the flow-static unidirectional case we have the equation $2g' - 2gr + sg = 0$ as well as the conditions $\sigma_2 = \sigma_3 = -\frac{1}{2\kappa}\left(-r' - s' + \frac{r^2}{2} + \frac{s^2}{2} + \frac{g^2}{2} + \frac{rs}{2}\right)$, $\sigma_1 = -\frac{1}{2\kappa}\left(-\frac{g^2}{2} + \frac{r^2}{2} + rs\right)$, $\lambda = 0$ and $\rho c^2 + \epsilon = -\frac{1}{2\kappa}\left(-2r' + \frac{3r^2}{2} - \frac{3g^2}{2}\right)$. A local metric spatial coordinate x exists with $x_{,a} = r_a$ and $r' = \frac{dr}{dx}$, $g' = \frac{dg}{dx}$, $s' = \frac{ds}{dx}$. The solution represents an extended mass sheet, stationary in time and rotating (if $g \neq 0$). If $g = 0$ the mass sheet is static. We can look for specific solutions now for a special equation of state, such as the ideal gas law. Thus we have a stationary unidirectional

cloud of gas, with density and pressure even functions of x alone that fall to zero as $x \to +\infty$ or $x \to -\infty$. All physical parameters depend only on x in this problem. For the material to behave as a classical ideal gas (for which we have all the explicit thermodynamic functions known) the thermal energy of the gas molecules must be much less than their rest energy, and $P \ll \rho c^2 + \epsilon$. The stationary unidirectional case is vastly different from the Robertson-Walker case (omnidirectional) where pressure seemed to have little effect. We shall see that a small but non-zero pressure is entirely different from the dust solutions considered in (III.21) and (III.22) because in this case a genuine mass sheet of finite width exists. Despite the fact that our gas is in the classical regime locally, if its mass is large enough we may expect to need relativistic gravitational theory.

Because the solution is stationary, it has time to reach equilibrium, and there is no problem with Fourier's Law of heat conduction (IV.5) so we may write the heat flux[1] as $q^a = \lambda v^a = -\xi\gamma^{ab}(\Theta_{,b} + \Theta\dot{u}_b)$ where ξ is the coefficient of thermal conductivity which is in general ≥ 0, and we assume here $\xi \neq 0$. For a stationary solution $\dot{\Theta} = -\frac{d\Theta}{d\tau}$ is zero, i.e. $\Theta_{,4} = -\dot{\Theta} = 0$ in frame components. Hence for zero heat flow, as we have here, $0 = \Theta_{,b} + \Theta\dot{u}_b$ so that $\Theta = \Theta_1 e^{-\Psi}$ where Ψ is the gravitational potential, $\Psi_{,a} = \dot{u}_a$. In particular Θ is a function only of Ψ (which is trivial here) and this applies also to the rigidly rotating perfect fluid case of (V.13). In that solution, P, $\rho c^2 + \epsilon$ and Θ are functions only of Ψ which suggests that the entire thermodynamic state may be a function only of Ψ, even though the local magnitude of gravitational acceleration is not.

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[1] Eckart, C. Phys. Rev. 58, 919(1940).

Returning now to the unidirectional case, we have for the monatomic ideal gas, $\epsilon = \frac{3}{2} K\Theta N$ where K is Boltzmann's constant, Θ is temperature, and $N = \frac{\rho}{m_0}$ is the number of particles per unit volume each having rest mass m_0. The equation of state is $P = \frac{\rho K\Theta}{m_0} = \frac{2}{3}\epsilon$, called the *ideal gas law*. We then also have

$$P = -\sigma_I = \frac{1}{2\kappa}\left(-\frac{g^2}{2}+\frac{r^2}{2}+rs\right) = \frac{1}{2\kappa}\left(-r' - s' + \frac{r^2}{2}+\frac{s^2}{2}+\frac{g^2}{2}+\frac{rs}{2}\right),$$

$$\rho = ab^2, \quad \Theta = \Theta_0 e^{-\Psi}, \quad \Psi' = -\frac{s}{2} = \frac{d\Psi}{dx}.$$

All functions depend only on x, in particular $\dot{a} = \dot{b} = \dot{\theta} = 0$. We may take $k = p = q = \hat{f} = h = 0$ in (III.15) so from (V.2.2), $w = v$, $\theta' = av$ and $b' = \frac{r}{2} b$. Also $\Theta' = -\Psi'\Theta = \frac{s}{2}\Theta$, $r' + s' = \frac{s^2}{2} + g^2 - \frac{rs}{2}$, $P = c_0(\rho c^2+\epsilon)$, $c_0 = \left(\frac{3}{2} + \frac{m_0 c^2}{K\Theta}\right)^{-1}$. We can show that from a physical point of view the solution exists uniquely. In analogy to the static extended mass sheet of (V.6) we can solve in the stationary case to obtain the generalization of (V.6.1) for $x > 0$ as

$$\mathfrak{z}' = \left(1+\frac{1}{c_0}\right)\frac{s\mathfrak{z}}{2}, \quad g' = -\frac{3sg}{2} - g\sqrt{s^2+2\mathfrak{z}+g^2},$$
$$s' = -\frac{1}{2}\left(3+\frac{1}{c_0}\right)\mathfrak{z} + g^2 - \frac{s^2}{2} - s\sqrt{s^2+2\mathfrak{z}+g^2}. \tag{V.19.1}$$

Here, $\mathfrak{z}$ represents mass density and pressure, $\mathfrak{z} = 2\kappa P = 2\kappa c_0(\rho c^2+\epsilon)$, s the acceleration ($\dot{u}^a = -\frac{s}{2} r^a$), and g the rotation or vorticity ($\upsilon^a = -\frac{g}{2} r^a$). As functions of x, $\mathfrak{z}$ and g are even and s is odd. If we specify the mass of a molecule (m_0), the value of $\mathfrak{z}$ at $x = 0$ (i.e. central pressure) the value[1] of Θ at $x = 0$ (i.e. central temperature) and the value of g at $x = 0$, then we can numerically solve the following equations for a unique solution:

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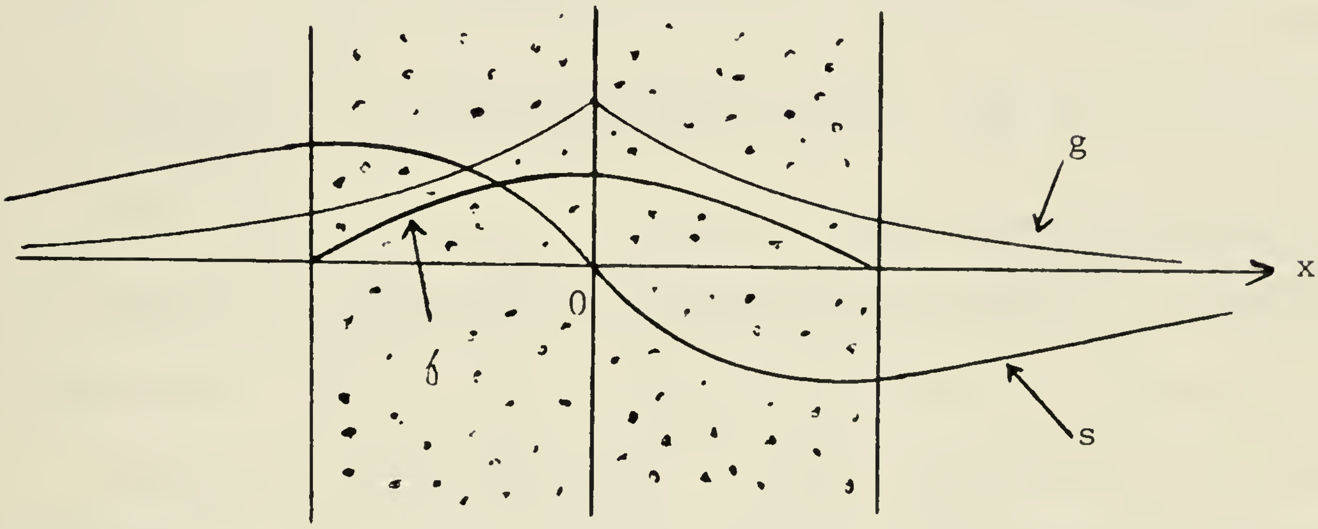
[1] The central temperature should satisfy $K\Theta << m_0c^2$.

$$\Theta' = \frac{s}{2}\,\Theta, \quad c_0 = \left(\frac{3}{2} + \frac{m_0 c^2}{K\Theta}\right)^{-1}, \quad \delta' = \left(1 + \frac{1}{c_0}\right)\frac{s\delta}{2},$$

$$g' = -\frac{3sg}{2} - g\sqrt{s^2 + 2\delta + g^2}\,, \tag{V.19.2}$$

$$s' = -\frac{1}{2}\left(3 + \frac{1}{c_0}\right)\delta + g^2 - \frac{s^2}{2} - s\sqrt{s^2 + 2\delta + g^2}, \quad x \geq 0.$$

Of course we take $s = 0$ at $x = 0$ by symmetry. If we



take $g(x=0) = 0$ then $g \equiv 0$ and the solution is static. Remember for $x > 0$, $s < 0$ so $\sqrt{s^2} = -s$. Assuming $g << s$ (slow rotation) we can take the exterior vacuum equations

$$g' = -\frac{3sg}{2} - g\sqrt{s^2 + g^2}\,, \quad s' = g^2 - \frac{s^2}{2} - s\sqrt{s^2 + g^2} \tag{V.19.3}$$

and approximate to get $s' = s^2/2$, $g' = -\frac{gs}{2}$ so that approximately $s = -\frac{2}{x}$, $g = c_1 x$, $c_1 \geq 0$ for constant c_1. Thus the approximation condition is violated for sufficiently large $x > 0$, if $c_1 \neq 0$. The equations (V.19.3) can be integrated if we put $g = \omega \cos\theta$, $s = \omega \sin\theta$, $\omega > 0$, $x > 0$, $g \geq 0$, $s < 0$, $-\frac{\pi}{2} \leq \theta \leq 0$. The vacuum equations then become

$$\theta' = \omega\cos\theta, \quad \omega' = -\left(1 + \frac{\sin\theta}{2}\right)\omega^2 \tag{V.19.4}$$

and we may put $\frac{d\omega}{d\theta} = \frac{\omega'}{\theta'}$ and separate variables, integrating to get $\omega = \frac{c_1(1-\sin\theta)}{\sqrt{\cos\theta}}$ where $c_1 > 0$ is constant. The important property is the relation $-\frac{1}{2}\omega^2 \geq \omega' \geq -\frac{3}{2}\omega^2$ which guarantees the typical

x^{-1} behavior at least, for large x.

We can see that the expression for ω as a function of θ gives us an unexpected result contrary to intuition in Newtonian theory. Since from (4), $\theta' \geq 0$ always we expect θ to increase as a function of x. In fact it breaks out of the range $-\frac{\pi}{2} \leq \theta \leq 0$ and into the range $0 \leq \theta \leq \frac{\pi}{2}$ approaching $\frac{\pi}{2}$ as $x \to +\infty$. This follows from $\omega = \dfrac{c_1(1-\sin\theta)}{\sqrt{\cos\theta}}$ and the fact that $c_1 > 0$ and $\omega \to 0$ as $x \to +\infty$. Thus for large enough x, s switches signs becoming positive and our stationary rest frame feels "gravitational repulsion". This phenomenon appears only when $g \neq 0$, i.e. the mass sheet is rotating.

Like $\mathfrak{z}$, the temperature Θ is an even function of x with a maximum at $x = 0$. With the functions Θ, $\mathfrak{z}$, g and s determined by solving (V.19.2) for $x > 0$ we can obtain $r = -s - \sqrt{s^2 + 2\mathfrak{z} + g^2}$ as a function of x (from $\mathfrak{z} = -\frac{g^2}{2} + \frac{r^2}{2} + rs$). We then specify $b(0) = b_0 > 0$ arbitrarily and solve for $b(x)$ using $b' = \frac{r}{2}b$, and $b(-x) = b(x)$ determines b for $x < 0$. We then determine ρ from $\rho = \dfrac{m_0\,\mathfrak{z}}{2\kappa K\Theta}$ as a function of x and obtain a from $a = \rho/b^2$. The values of θ and $v = w$ in (V.2.2), (not the ω in (V.19.4)), can be obtained with $\theta' = av$ in a number of ways that are not of direct interest to the physics of the problem. We see that the thermodynamics, equations of state and thermal conductivity determine the solution physically for the relativistic gravitational problem of the stationary unidirectional classical ideal gas. The fact that (V.2.2) imposes no direct condition on a' was important for this solution to work. It is also clear that (V.3.1) holds as well. The ideal gas is thermoelastic in the sense of (IV.10), and the motion is rigid $\theta_{ab} = 0$ and therefore expansion and shear free, i.e. $f = t = 0$ in unidirectional notation.

Even though there is no temporal expansion $(\theta = 0)$ or temporal shearing $(\sigma_{ab} = 0)$ we do have spatial expansion and shear even in the static case $(g = 0)$. From (V.4) we have the orthogonalized generalized stretch given by

$$\theta_{abc} = \left(\frac{r}{2} - \frac{a'}{a}\right) r_a r_b r_c - \frac{r}{2} \gamma_{ab} r_c \qquad \text{(V.19.4)}$$

which is non-zero in our case here, so along r^c a spatial deformation $\theta_{abc} r^c$ exists.

Of course the entropy density η for the monatomic ideal gas from the classical Sackur-Tetrode[1] equation is

$$\eta = \frac{K\rho}{m_0}\left[\frac{5}{2} + \ell n\left[\left(\frac{m_0 K\Theta}{2\pi\hbar^2}\right)^{3/2} \frac{m_0}{\rho}\right]\right] \qquad \text{(V.19.5)}$$

where $\hbar$ is Planck's constant. We see that the entropy flux vector is $S^a = \eta u^a$ here since $q^a = \lambda v^a = 0$, (cf. (IV.8)). In this stationary (equilibrium) configuration $S^a{}_{;a} = 0$ since $\eta_{,4} = 0$ in frame components follows from $\Theta_{,4} = 0$ and $\rho_{,4} = 0$ using (V.19.5).

(V.20) Spherical and Rotating Cases (Degenerate Fermi Gas)

The rigidly rotating perfect fluid is of great interest in astrophysics. It has been shown[2] that a stationary star consisting of a viscous heat conducting general relativistic fluid must be axisymmetric with rigid motion and zero heat flow. Thus its energy tensor is of the perfect fluid form and the Kerr type Ricci coefficients for rigid motion are appropriate (V.16). We saw in (V.13) that P and $\rho c^2 + \epsilon$ were functions only of Ψ, the gravitational potential, with

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[1] Kittel, Charles, Thermal Physics, John Wiley & Sons, N.Y. (1969), p. 167.

[2] Lindblom, Lee, Astrophysical J. 208, (1976), 873-880.

$\frac{d\delta}{d\Psi} = -\left(1+\frac{1}{c_0}\right)\delta$, $P = \frac{\delta}{2\kappa} = c_0(\rho c^2+\epsilon)$. Also in (V.19) we saw $\frac{d\Theta}{d\Psi} = -\Theta$.
The results in (V.13) can be seen as a consequence of the following.

Theorem: *In a space-time with a perfect fluid undergoing an isochoric motion with a gravitational potential* Ψ *existing and for which the pressure satisfies* $P_{,a}u^a = 0$, *the pressure is a function only of* Ψ.

Proof: We write $T^{ab} = -(\rho c^2+\epsilon)u^a u^b - P\gamma^{ab}$ and substitute into $T^{ab}{}_{;b} = 0$ to obtain

$$0 = (\rho c^2+\epsilon)_{,b}u^b u^a + (\rho c^2+\epsilon)\dot{u}^a + (\rho c^2+\epsilon)\theta u^a$$

$$+ P_{,b}\gamma^{ab} + P(\dot{u}^a + u^a\theta). \tag{V.20.1}$$

We multiply by u_a using $u^a u_a = -1$ to get

$$0 = (\rho c^2+\epsilon)_{,b}u^b + (\rho c^2+\epsilon+P)\theta.$$

For isochoric motion $\theta = 0$ so $(\rho c^2+\epsilon)_{,b}u^b = 0$ so substituting this and $\theta = 0$ into (V.20.1) we have

$$0 = (\rho c^2+\epsilon+P)\dot{u}^a + P_{,b}\gamma^{ab}.$$

Now using $P_{,a}u^a = 0$ we may write $P_{,b} = -(\rho c^2+\epsilon+P)\dot{u}_b$ and since $\dot{u}_b = \Psi_{,b}$ we find that on the hypersurface of constant Ψ, P does does not change. Hence P is a function only of Ψ.

Corollary: *For a stationary space-time with a perfect fluid undergoing a motion with the flow vector parallel to a time-like Killing vector, the pressure is a function only of the gravitational potential.*

Proof: Here $\theta = 0$ since $\theta_{ab} = 0$ and Ψ exists with $e^{\Psi}u^a$ a time-like Killing vector, and $P_{,4} = 0$ in adapted frame components for the

flow-static Ricci coefficients. Thus the theorem applies.

A constitutive equation that is easy to state and is of interest in astrophysics for white dwarfs and other dense, relatively cool stars is the one for a degenerate Fermi gas[1]. The condition for degeneracy is $K\Theta << e_f$ where e_f is the Fermi energy of one particle given by $e_f = \frac{\hbar^2}{2m_0}\left(3\pi^2 \frac{\rho}{m_0}\right)^{2/3}$. The energy density is $\epsilon = \frac{3}{10}\frac{\hbar^2}{m_0}(3\pi^2)^{2/3}\left(\frac{\rho}{m_0}\right)^{5/3} = \frac{3}{2}P$ and so c_0 is obtained from $c_0 = \left(\frac{3}{2} + \frac{5m_0^{8/3}c^2}{(3\pi^2)^{2/3}\hbar^2\rho^{2/3}}\right)^{-1}$. Although it satisfies $\frac{d\Theta}{d\Psi} = -\Theta$, the temperature Θ is much less than the Fermi energy at every point in the solution of our axisymmetric stationary rotating star, and this is assumed for the validity of the above equations.

Since $\mathfrak{z}$ is a constant multiple of $\rho^{5/3}$ we have $\frac{d}{d\Psi}(\rho^{5/3}) = -\left(1+\frac{1}{c_0}\right)\rho^{5/3}$ and hence $\frac{d\rho}{d\Psi} = -\frac{3}{2}\rho - \frac{3m_0^{8/3}c^2\rho^{1/3}}{(3\pi^2)^{2/3}\hbar^2}$ which can be integrated to give us the density ρ as a function of the gravitational potential Ψ. In the spherical case, the complete solution may be obtained using (V.7.5).

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[1] Charles Kittel, Thermal Physics, Wiley, N.Y. (1969), p. 225-230.

(V.21) Cylindrical Symmetry - Junction Conditions.

In this section we shall consider solving Einstein's equations for an exterior vacuum solution, and internally for a material medium and matching the solutions across the bounding surface. If the hypersurface S between matter and vacuum is given by the equation $\phi = 0$, the Lichnerowicz junction conditions[1] tell us that the Einstein tensor G^{ij} (equivalently the energy momentum tensor T^{ij}) may have a finite discontinuity at points of S. However $G^{ij}\phi_{,j}$ (or $T^{ij}\phi_{,j}$) must be continuous at S for all admissible coordinates. These are the coordinate systems for which g_{ij} is C^1 and piecewise C^3 which is assumed for obtaining these junction conditions. For a perfect fluid this means that the pressure P must be continuous (and thus have value zero) on S while $\rho c^2 + \epsilon$ may or may not be continuous on S, depending on the constitutive equation, or equation of state.

It is of interest to apply these conditions in frame components, which we will do in this section for a cylindrical stationary perfect fluid. Our admissible frame components will have $T_b{}^a{}_c$ continuous with possible finite jump discontinuities at S for the derivatives $T_b{}^a{}_{c,d}$.

Let us consider a cylinder of infinite length whose axis of symmetry is the z axis and which is undergoing a stationary rotation about this axis. We write the metric for this case as

$$ds^2 = -e^{2\Psi(r)}dt^2 + F(r)dz^2 + G(r)dr^2 + H(r)r^2d\theta^2 + K(r)dtd\theta.$$

We transform into flow static frame components with $v^1_{(1)}$, $v^2_{(2)}$, $v^3_{(3)}$, $v^4_{(4)}$, $v^4_{(2)}$ the only nonzero transition coefficients so that only $T_2{}^2{}_1$,

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[1] Lichnerowicz, A., Théories Relativistes de la Gravitation et de l'Électromagnétisme, Masson et Cie, Paris (1955).

$T_3{}^3{}_1$, $T_1{}^4{}_2$ and $T_4{}^4{}_1$ are nonzero and they have derivatives only in the 1 direction. We write ($\mathcal{s}_{,\ell} = \mathcal{s}' r_\ell$, $g_{,\ell} = g' r_\ell$ etc.)

$$T_b{}^a{}_c = \mathcal{s} s^a r_{[b} s_{c]} + g t^a t_{[b} r_{c]} + \mathcal{t} u^a r_{[b} s_{c]} + \mathcal{k} u^a u_{[b} r_{c]} .$$

Here the vorticity is $\upsilon^a = -\frac{\mathcal{t}}{4} t^a$, the deformation rate $\theta_{ab} = 0$ and the acceleration $\dot{u}^a = \frac{\mathcal{k}}{2} r^a$. The Ricci coefficients are strongly flow static and weakly directionally invariant along s^a and t^a. Since $T_a{}^1{}_b = 0$ we have the existence of a local metric radial coordinate x with $x_{,a} = r_a$, x being the distance from the z axis (or axis of symmetry). The principal stresses and mass and energy density are

$$\sigma_1 = \frac{1}{2\kappa}\left(\frac{\mathcal{k}g}{2} - \frac{\mathcal{k}\mathcal{s}}{2} - \frac{\mathcal{t}^2}{8} + \frac{g\mathcal{s}}{2}\right) ,$$

$$\sigma_2 = \frac{1}{2\kappa}\left(g' - \mathcal{k}' - \frac{\mathcal{k}^2}{2} + \frac{\mathcal{k}g}{2} - \frac{\mathcal{t}^2}{8} - \frac{g^2}{2}\right) ,$$

$$\sigma_3 = \frac{1}{2\kappa}\left(-\mathcal{s}' - \mathcal{k}' - \frac{\mathcal{k}^2}{2} - \frac{\mathcal{k}\mathcal{s}}{2} + \frac{\mathcal{t}^2}{8} - \frac{\mathcal{s}^2}{2}\right) ,$$

$$\rho c^2 + \varepsilon = \frac{1}{2\kappa}\left(-\mathcal{s}' + g' + \frac{3}{8}\mathcal{t}^2 - \frac{g^2}{2} + \frac{g\mathcal{s}}{2} - \frac{\mathcal{s}^2}{2}\right) .$$

The Jacobi identity and integrability conditions are trivial and the zero heat flux condition (in s^a direction) gives us

$$0 = -2\mathcal{t}' - 2\mathcal{k}\mathcal{t} + \mathcal{t}g .$$

For a perfect fluid we put $\sigma_1 = \sigma_2 = \sigma_3 = -P = -\mathfrak{f}/2\kappa$ and $\rho c^2 + \varepsilon = P/c_0$. This then implies that

$$\mathcal{t}' = -\mathcal{k}\mathcal{t} + \frac{\mathcal{t}g}{2} ,$$

$$\mathfrak{f} = -\frac{\mathcal{k}g}{2} + \frac{\mathcal{k}\mathcal{s}}{2} + \frac{\mathcal{t}^2}{8} - \frac{g\mathcal{s}}{2} ,$$

$$g' = \frac{\mathfrak{f}}{2c_0} - \frac{\mathcal{t}^2}{16} + \frac{g^2}{2} - \frac{g\mathcal{s}}{4} - \frac{\mathcal{k}\mathcal{s}}{4} - \frac{\mathcal{k}g}{4} ,$$

$$\mathcal{s}' = -\frac{\mathfrak{f}}{2c_0} + \frac{5}{16}\mathcal{t}^2 - \frac{\mathcal{s}^2}{2} - \frac{\mathcal{k}g}{4} - \frac{\mathcal{k}\mathcal{s}}{4} + \frac{g\mathcal{s}}{4} ,$$

$$\mathcal{k}' = \mathfrak{f} + \frac{\mathfrak{f}}{2c_0} - \frac{3}{16}\mathcal{t}^2 - \frac{g\mathcal{s}}{4} - \frac{\mathcal{k}\mathcal{s}}{4} + \frac{\mathcal{k}g}{4} - \frac{\mathcal{k}^2}{2} ,$$

from which we can show that $f' = -\left(\frac{1+c_0}{2c_0}\right) kf$.

We can solve for t^2 to get $\frac{t^2}{16} = \frac{f}{2} + \frac{kg}{4} + \frac{gs}{4} - \frac{ks}{4}$ and eliminate t^2 to get the system of equations

$$
\begin{aligned}
g' &= \frac{f}{2c_0} - \frac{f}{2} + \frac{g^2}{2} - \frac{kg}{2} - \frac{gs}{2}\,, \\
s' &= \frac{5f}{2} - \frac{f}{2c_0} - \frac{s^2}{2} + kg + \frac{3}{2}\,gs - \frac{3}{2}\,ks, \\
k' &= \frac{f}{2c_0} - \frac{f}{2} - \frac{k^2}{2} - \frac{kg}{2} - gs + \frac{ks}{2}\,, \\
f' &= -\left(\frac{1+c_0}{2c_0}\right) kf.
\end{aligned}
\tag{V.21.1}
$$

In the case of a vacuum solution, we put $f = 0$ to get

$$
\begin{aligned}
g' &= \frac{g^2}{2} - \frac{kg}{2} - \frac{gs}{2}\,, \\
s' &= -\frac{s^2}{2} + kg + \frac{3}{2}\,gs - \frac{3}{2}\,ks, \\
k' &= -\frac{k^2}{2} - \frac{kg}{2} - gs + \frac{ks}{2}\,.
\end{aligned}
\tag{V.21.2}
$$

Since $\frac{g'}{g} = (\ell n\, g)' = \frac{1}{2}\,(g-k-s)$ we see that $k + s = g - 2(\ell n\, g)'$. But $k' + s' = -\frac{1}{2}\,(s+k)^2 + \frac{g}{2}\,(s+k)$ so if we let $w = k + s$ we find $w = g - 2(\ell n\, g)'$ and $w' = -\frac{1}{2}\,w^2 + \frac{g}{2}\,w$. Hence $(\ell n\, g)' = \frac{1}{2}\,(g-w)$ and $(\ell n\, w)' = \frac{1}{2}\,(g-w)$ so that $\ell n \left(\frac{g}{w}\right)$ is a constant. Let $w = k_0 g$, k_0 a constant, so $(\ell n\, g)' = \left(\frac{1-k_0}{2}\right) g$ which implies $g = \frac{-2}{(1-k_0)x + c_1}$ for constant c_1. Here x is the local metric radial coordinate. By symmetry the singularity must occur at $x = 0$ so $c_1 = 0$ and so $g = \frac{2}{(k_0-1)x}$ and $w = k_0 g = \frac{2k_0}{(k_0-1)x}$. Since $s = w - k$ we write the differential equation for k as

$$
k' = -k^2 + \frac{(1+k_0)k}{(k_0-1)x} - \frac{4k_0}{(k_0-1)^2 x^2}\,. \tag{V.21.3}
$$

If we look for a solution of the form $k = \frac{k_1}{x}$ where k_1 is a constant,

and $k_1 > 0$ on physical grounds, then the implicit relation

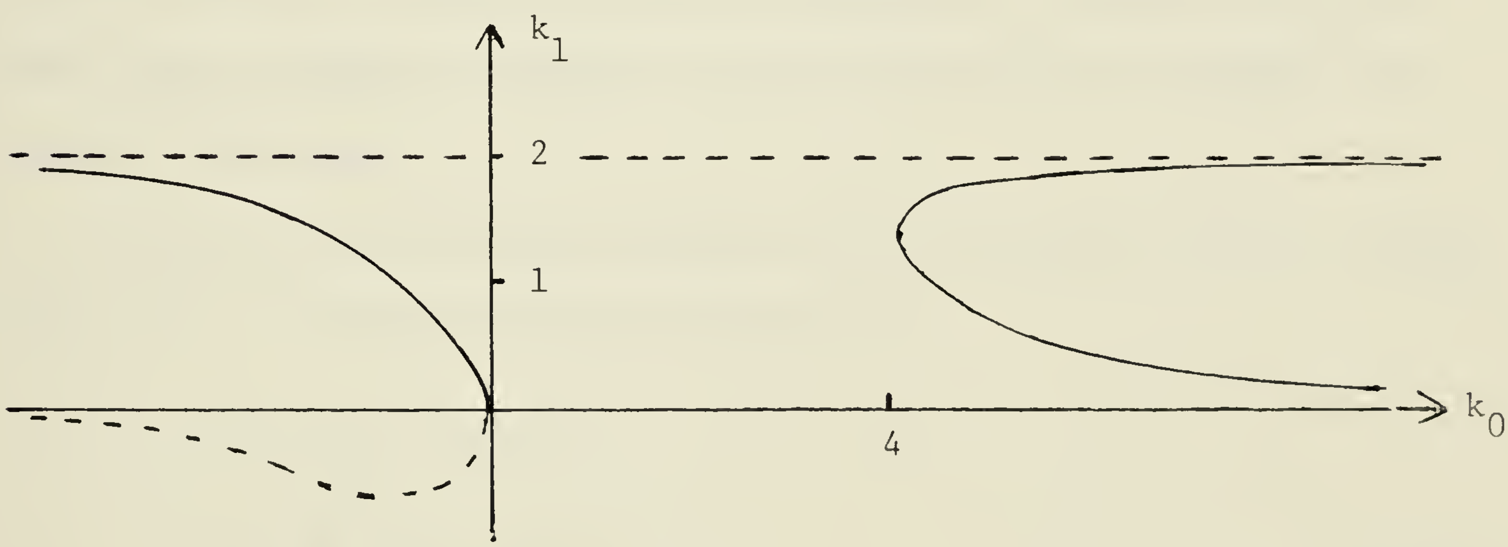
$$k_1(k_1-1) = \frac{(1+k_0)k_1}{k_0-1} - \frac{4k_0}{(k_0-1)^2}$$ must be satisfied, which we can solve

for k_1 to obtain

$$k_1 = \frac{k_0 \pm \sqrt{k_0^2 - 4k_0}}{k_0 - 1} \quad . \tag{V.21.4}$$

Another condition which must be true for a physical solution is $\frac{\mathscr{t}^2}{4} = \mathscr{k}g + g\mathscr{s} - \mathscr{k}\mathscr{s} \geq 0$. We find that $\mathscr{t} = 0$ for the exterior solution as can easily be verified. The boundary conditions imply that the interior is free of vorticity since $\mathscr{t}$ must be continuous at the transition surface. The differential equation for $\mathscr{t}$ then says $\mathscr{t}' = 0$ on the surface.

If we sketch a graph of (V.21.4) it will look like the following:



Only the positive values of k_1 are physically allowable. As $k_0 \to \infty$ we obtain the solution $\mathscr{k} = \frac{2}{x}$, $\mathscr{t} = g = \mathscr{s} = 0$. The solutions here in this graph are the zero vorticity ones. To obtain non-zero $\mathscr{t}$ solutions with rotation, we need the solutions of (V.21.3) which are not of the form $\frac{k_1}{x}$.

We can recast (V.21.3) in the form

$$x\,\frac{d(\mathscr{k}x)}{dx} = \frac{2k_0}{k_0 - 1}\,\mathscr{k}x - \mathscr{k}^2x^2 - \frac{4k_0}{(k_0-1)^2} \,,$$

which permits separation of variables. From the integration of $\frac{du}{(u-k_1^+)(u-k_1^-)} = -\frac{dx}{x}$ where $u = kx$ and $k_1^{\pm}$ are given by (V.21.4) we have

$$k = \left(\frac{1}{x}\right) \frac{k_1^-\left(\frac{c}{x}\right)^{k_1^+} - k_1^+\left(\frac{c}{x}\right)^{k_1^-}}{\left(\frac{c}{x}\right)^{k_1^+} - \left(\frac{c}{x}\right)^{k_1^-}}, \qquad \text{(V.21.5)}$$

for some constant of integration $c > 0$.

If we take the special limiting case as $k_0 \to \infty$ then $k_1^+ \to 2$ and $k_1^- \to 0$ so we obtain the functions

$$k = -\frac{2x}{c^2 - x^2}, \quad g = 0, \quad s = \frac{2c^2}{x(c^2-x^2)}, \quad t = \frac{\pm 4c}{c^2 - x^2}$$

as a particular vacuum solution, c providing a measurement of vorticity.

We have effectively solved the vacuum case entirely now for the original metric, i.e. all stationary cylindrical solutions. Now we would like to examine an explicit interior solution and consider the boundary conditions.

If we look for interior solutions with no rotation $t = 0$, and put $v = g - k - s$ we may write the equations as,

$$v' = \frac{\delta}{2c_0} - \frac{5\delta}{2} + \frac{v^2}{2},$$

$$k' = \frac{3\delta}{2} + \frac{\delta}{2c_0} + \frac{k}{2} v, \qquad \text{(V.21.6)}$$

$$\delta' = -\frac{k\delta}{2} - \frac{k\delta}{2c_0}.$$

Putting $v = \frac{-2}{x}$ so $v' = \frac{v^2}{2}$ we obtain the constitutive equation $c_0 = \frac{1}{5}$ not entirely unreasonable, as well as $\frac{dk}{dx} = 4\delta - \frac{k}{x}$, $\frac{d\delta}{dx} = -3k\delta$. This pair can be divided out and integrated to get $k = \frac{4x}{3(b^2+x^2)}$ and $\delta = \frac{2b^2}{3(b^2+x^2)^2}$. There is another solution with v identically zero that has the undesirable property of having k approach a non-zero positive

value as $x \to \infty$. Thus $v = -\frac{2}{x}$ is the only solution with appropriate reflection symmetries that gives a physically reasonable solution, with $\kappa(0) = 0$.

We then use $g' = \frac{\zeta}{2c_0} - \frac{\zeta}{2} + \frac{gv}{2}$ to obtain g as $g = \frac{2x}{3(b^2+x^2)}$ and hence $s = \frac{2}{x} - \frac{2x}{3(b^2+x^2)}$. For large x this solution approximates the vacuum solution $\kappa = \frac{4}{3x}$, $g = \frac{2}{3x}$, $s = \frac{4}{3x}$ which corresponds to $k_0 = 4$ and $k_1 = \frac{4}{3}$. In this case we do not actually have to impose boundary conditions as we would for an incompressible fluid with constant density. If we put $\frac{\zeta}{2c_0} = \frac{E}{2}$ a constant, while ζ is a function of position x then the hydrostatic pressure equation can be integrated to give $\zeta = -E + Ke^{-\Psi}$ where $K > E > 0$ is constant, and $\Psi = \int_0^x \frac{\kappa}{2}\,dx$ is the gravitational potential with $\kappa(0) = 0$. For large x, κ behaves like $\frac{k_1}{x}$ so Ψ increases logarithmetically to ∞ as $x \to \infty$. Since ζ is decreasing, there must exist x_0 for which $\zeta(x_0) = 0$, and it is at x_0 where the values of the Ricci coefficients are made to correspond to those for the vacuum exterior at x_0. For the cylindrical static case, (V.21.6) becomes

$$\Psi' = \frac{\kappa}{2}$$

$$v' = 3E - \frac{5}{2}Ke^{-\Psi} + \frac{v^2}{2}$$

$$\kappa' = -E + \frac{3}{2}Ke^{-\Psi} + \frac{\kappa}{2}v$$

where v behaves like $-\frac{2}{x}$ near $x = 0$ and $\kappa(0) = 0$, $\Psi(0) = 0$, $\kappa'(0) = \frac{3}{4}K - \frac{E}{2}$. These equations could be integrated numerically beginning at $x = 0$ for given values of K and E. The process is continued until $x = x_0$ where $\zeta = 0$. At that point the matter stops, and for $x > x_0$ we take the vacuum solution whose Ricci rotation

coefficients at x_0 equal those for the interior at x_0. This incorporates the boundary conditions of Lichnerowicz which state that T^{ab} may have a finite discontinuity across the transition surface (or hypersurface) but $T^{ab}n_b$ must be continuous at the transition, where n_b is the normal to the surface.

We can have boundary conditions and a transition surface even in cases where the density is not constant, under a more realistic constitutive equation. If in (V.21.6) we put $\frac{\mathfrak{p}}{2c_0} - \frac{5\mathfrak{p}}{2} = \frac{k^2}{2}$ a positive constant, then we may integrate to obtain $v = -k \cot\left(\frac{kx}{2}\right)$ and $\mathfrak{p} = -\frac{k^2}{6} + Ce^{-6\Psi}$ where $C > 0$ is constant. In this case we obtain

$$\mathfrak{k}' = 4Ce^{-6\Psi} - \frac{k^2}{6} - \frac{\mathfrak{k}k}{2} \cot\left(\frac{kx}{2}\right) \quad \text{and} \quad \Psi' = \frac{\mathfrak{k}}{2} .$$

Again $\mathfrak{k}(0) = \Psi(0) = 0$. Because of the $-\frac{k^2}{6}$ term in the expression for $\mathfrak{p}$, we see that the pressure $\mathfrak{p}$ goes to zero when the density $\frac{\mathfrak{p}}{c_0}$ is non-zero.

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